# Non-linear Vibration Harvesting

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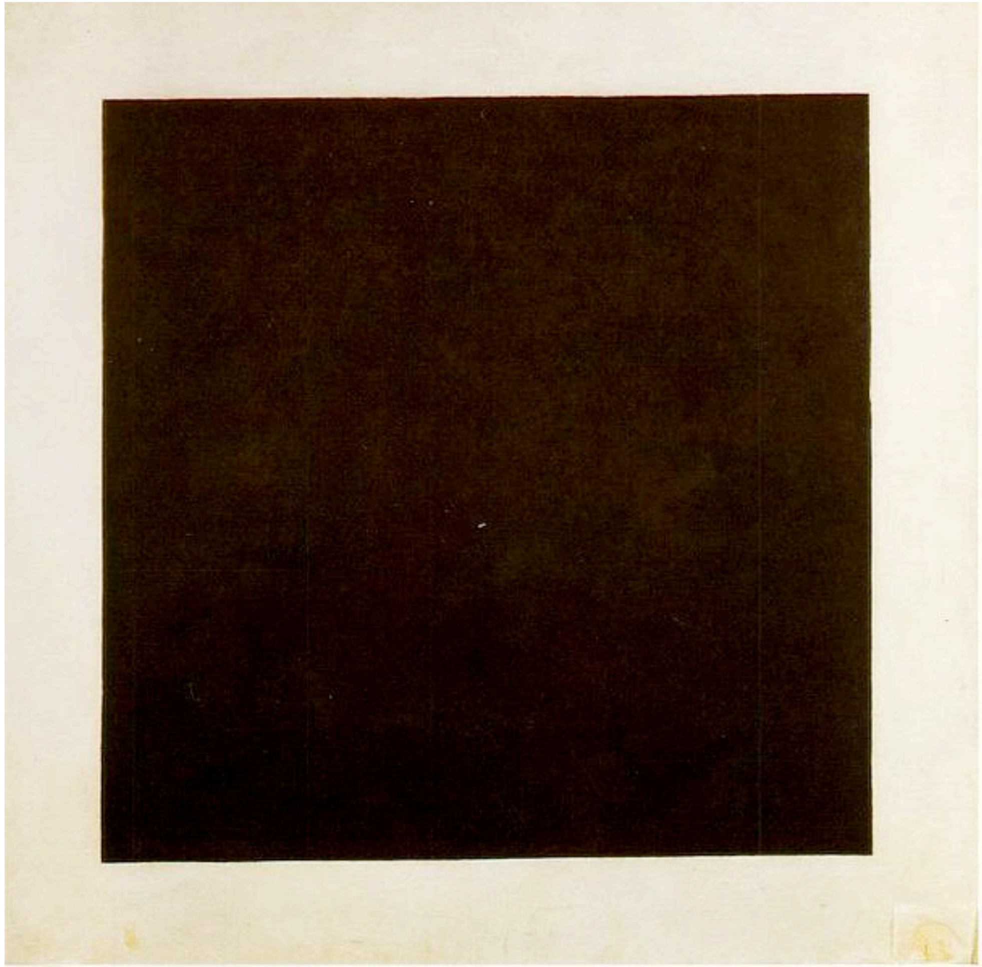
Noise in Physical Systems

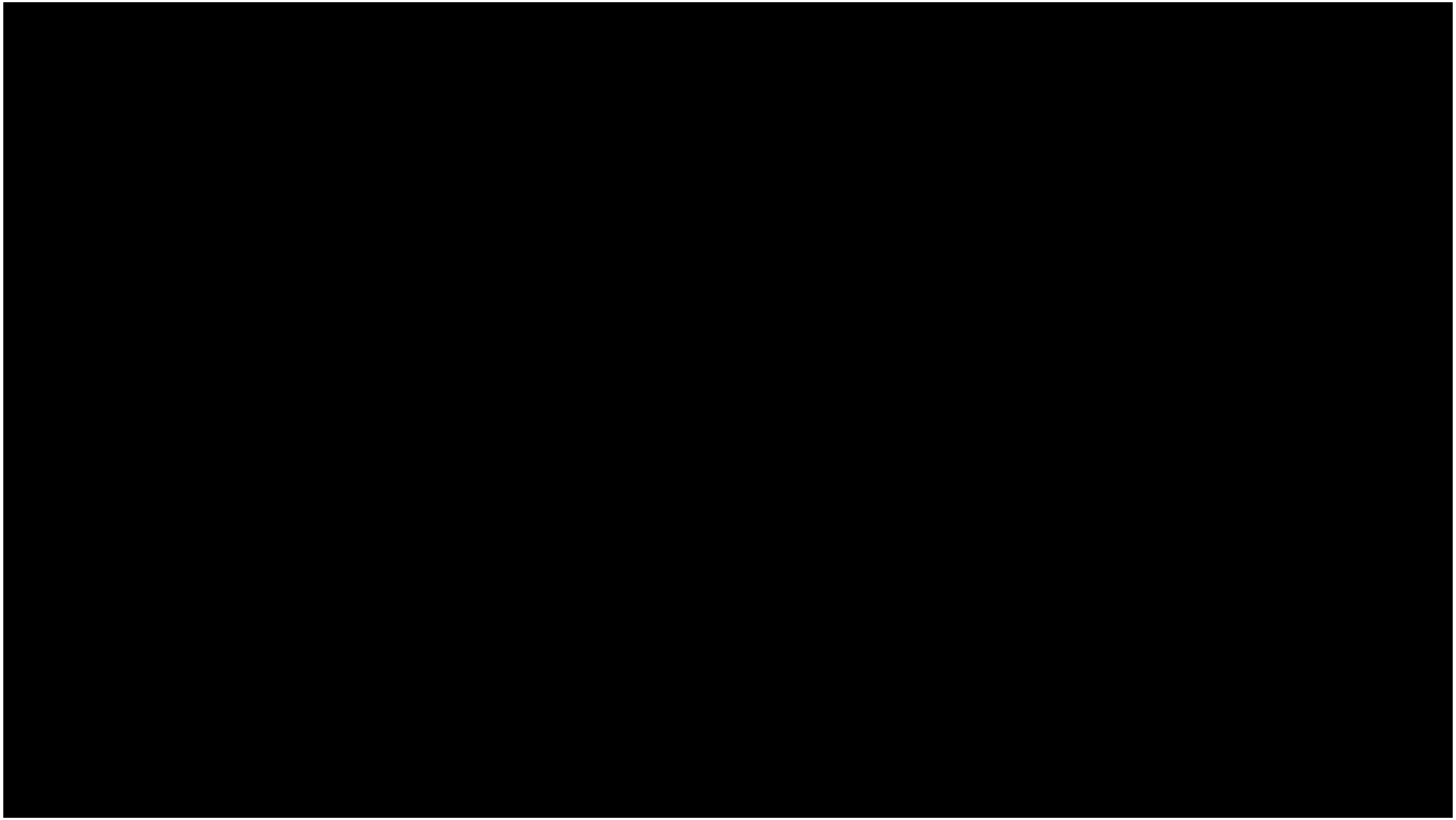
# Let’s start with some music:

(from Chopin to ‘adiabatic’ music…)

# From Caravaggio to Malevich:



# Back on Earth: from arts to an energy harvesting application



# Vibration energy harvesting

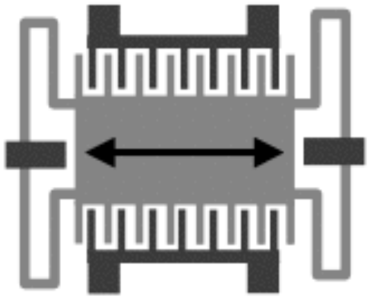
Four main transduction mechanisms

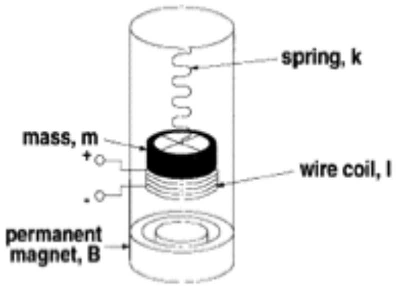
**Capacitive**: geometrical variations induce voltage difference

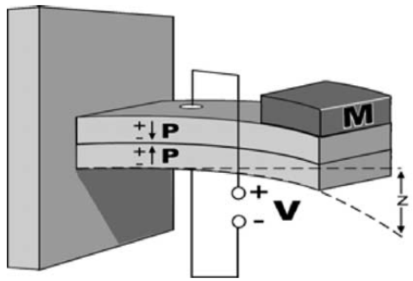
**Piezoelectric**: dynamical strain is converted into voltage difference.

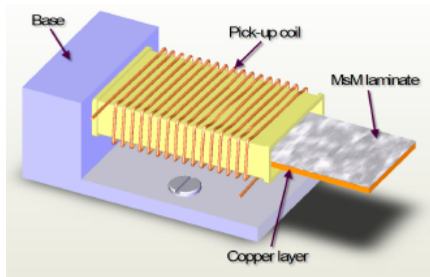
**Inductive**: dynamical oscillations of magnets induce electric current in coils

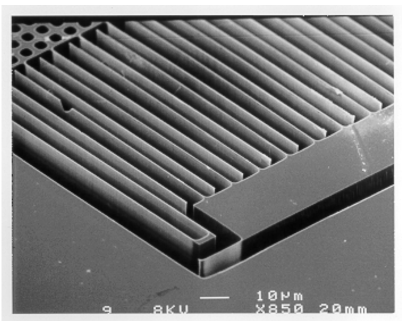
**Magnetostrictive:** stress produces a variable magnetic field that induces a current in an adjacent conductive coil.

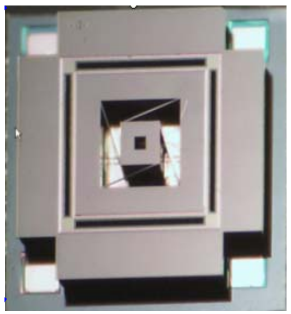


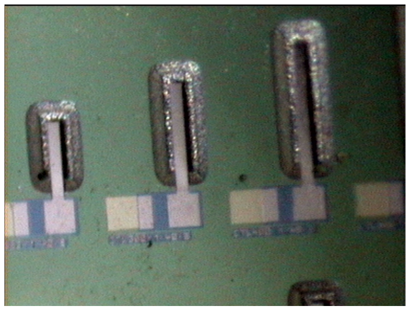


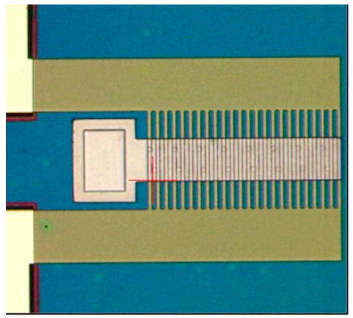



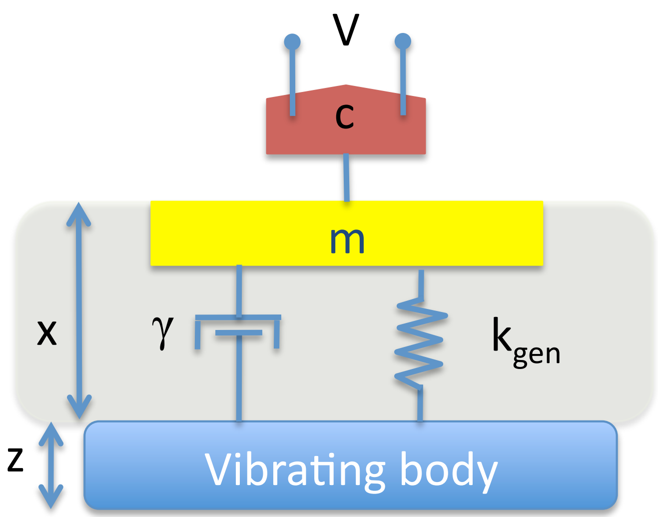









# Vibrations harvesting: the model




$$m\ddot{x} = -\frac{dU(x)}{dx} - \gamma\dot{x} - c(x,V) + \zeta_z$$

Force acting on harvester oscillator

Force due to the energy stored

Dissipative force

Reaction force due to the transduction mechanism

Input force

$$\begin{cases} m\ddot{x} = -\frac{dU(x)}{dx} - \gamma\dot{x} - c(x,V) + \zeta_z \\ \dot{V} = F(\dot{x},V) \end{cases}$$

Details depend on the physics…

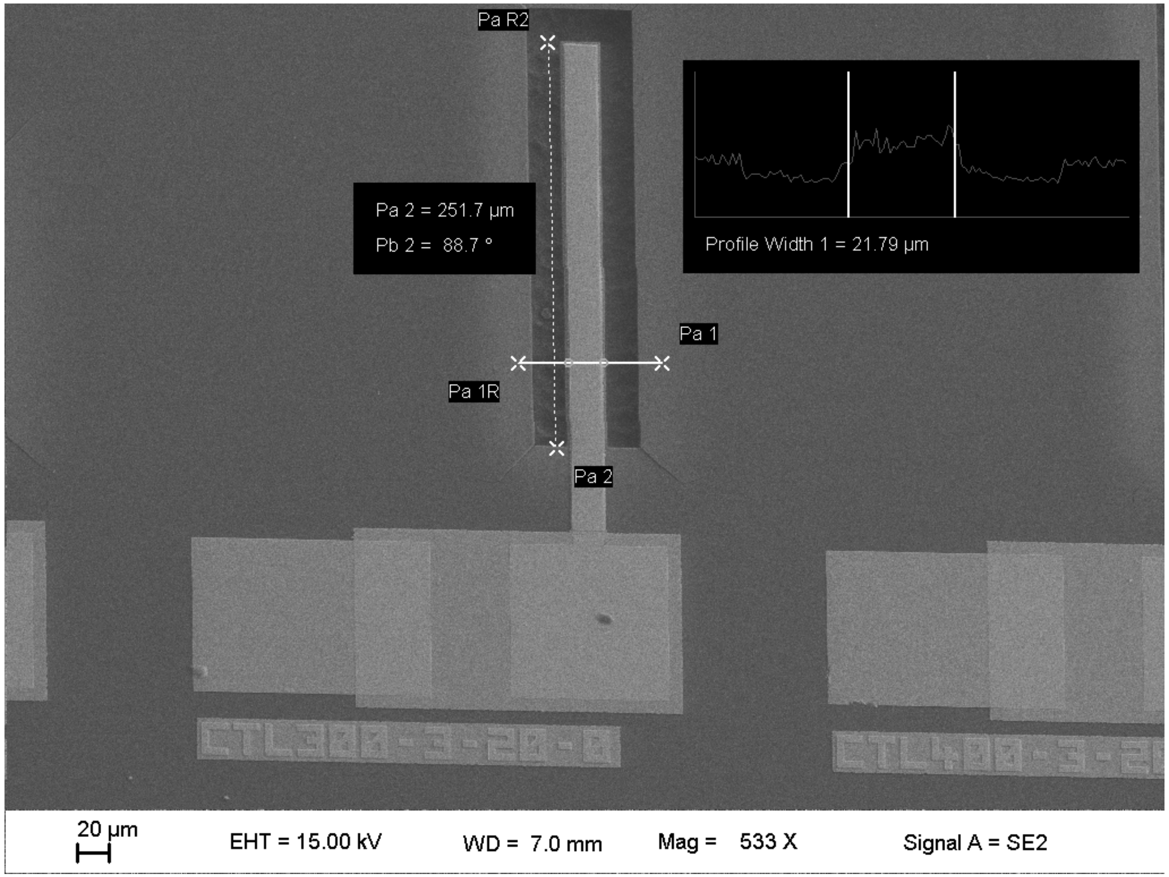
Equations that link the vibration-induced displacement with the Voltage

# Vibrations harvesting: the transduction mechanism

We will focus on **Piezoelectricity** because for practical reasons has the best coupling factor.

**Capacitive**: is more easy to scale down but you have to pay a debt: it needs a bias voltage

**Inductive and Magnetostrictive**: are more difficult to be scaled down and have a lower coupling factor



# Vibrations harvesting: the model for piezo

$$
\begin{cases}
m\ddot{x} = -\dfrac{dU(x)}{dx} - \gamma\dot{x} - K_v V + \zeta_z \\
\dot{V} = K_c \dot{x} - \dfrac{1}{\tau_p} V
\end{cases}
$$

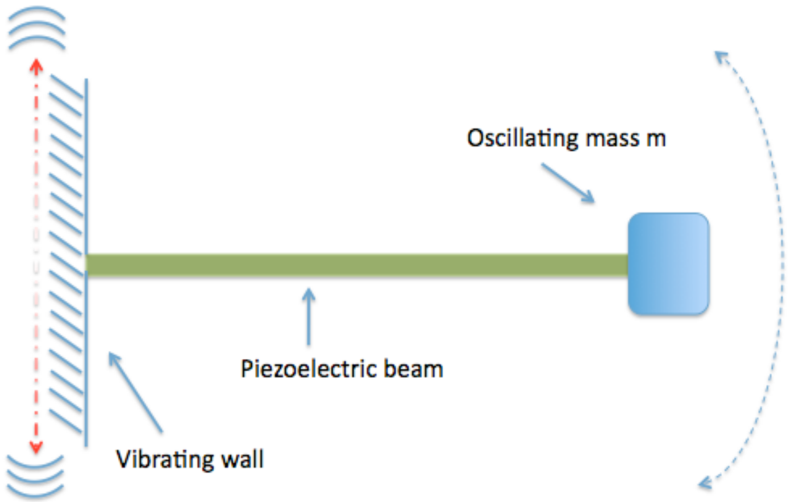
The Physics of piezo materials

That for a beam are:

$$K_v = \frac{K_{eff} d_{31} a}{2 t_p k_1}$$

$$K_c = \frac{t_p d_{31} Y_p^E k_1}{a \varepsilon_p}$$

Now we focus on $\zeta_z$



# The random character of kinetic energy

$\zeta_z$ Represents the vibrational stochastic force

**Random vibrations / noise**

**Thermal noise (NOT POSSIBLE AT EQUILIBRIUM!!!)**
**Acoustic noise**
**Seismic noise**
**Ambient noise** (wind, pressure fluctuations, …)
**Man made vibrations** (human motion, machine vibrations,…)

All different for intensity, spectrum, statistics

How can we harvest them ?

# Linear system

If a linear system is considered: $U(x) \approx x^2$

1) There exist a simple math theory to solve the equations
2) They have a resonant behaviour (resonance frequency)
3) They can be "easily" realized with cantilevers and pendula

# Linear system

A general framework for ODE described by:

$$\dot{x}_1 = f_1(x_1, \cdots, x_n)$$
$$\vdots$$
$$\dot{x}_n = f_n(x_1, \cdots, x_n)$$

Is linear if all the $x_i$ on the right hand side appear to the first power only.

The system described by: $$m\ddot{x} = -\frac{dU(x)}{dx} - \gamma\dot{x} - K_V V + \zeta_z$$ with $U(x) = \frac{1}{2}kx^2$

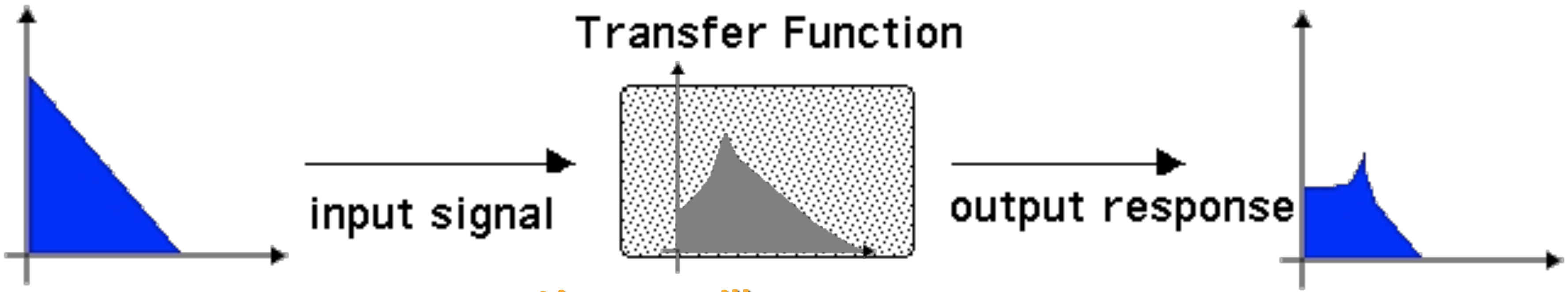
$$m\ddot{x} = -kx - \gamma\dot{x} - K_V V + \zeta_z$$

Can be rewritten as:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \tfrac{1}{m}(-kx_1 - \gamma x_2 - K_V V + \zeta_z)$$

# Linear system

***X(s)*** i.e. ambient energy ***H(s)*** ***Y(s)*** i.e. output energy




**Linear oscillator**

The transfer function is a math function of the frequency, in the complex domain, that can be used to represent the performance of a linear system and can act as a filter…

$$Y(s) = H(s)\,X(s)$$

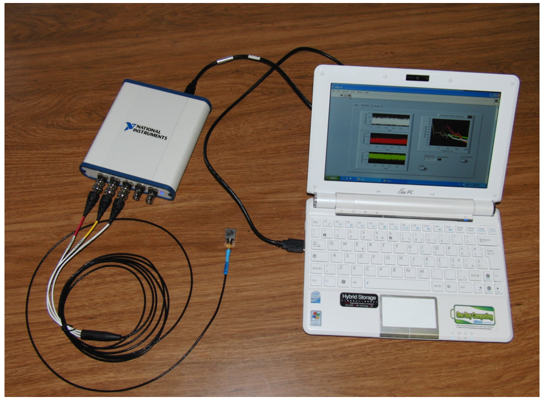
For a linear system the transfer function presents one or more peeks corresponding to the resonace frequencies.

A linear system is the most performing if its resonance frequency is where the incoming energy is abundant…

**This is a serious limitation when you want to build a small energy harvesting system working in a real environment…**

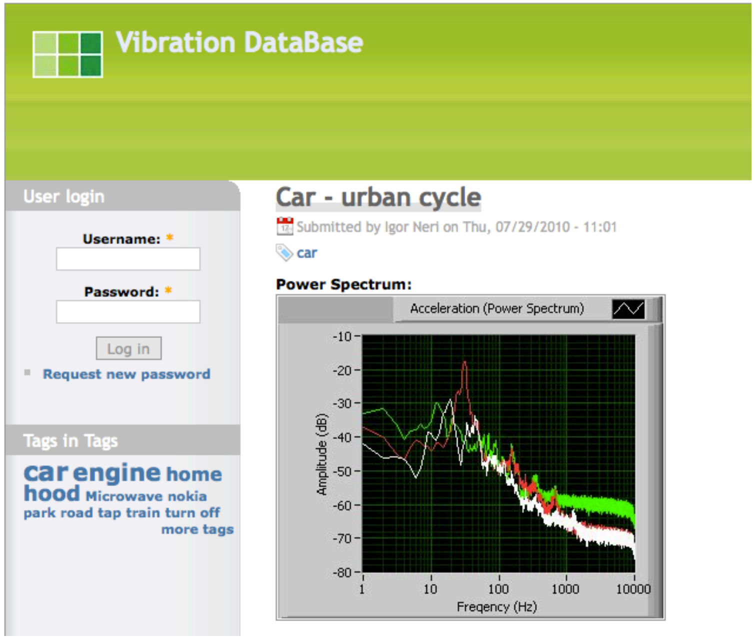
## For two main reasons:

The frequency spectrum of available vibrations instead of being sharply peaked at some frequency is usually very broad.

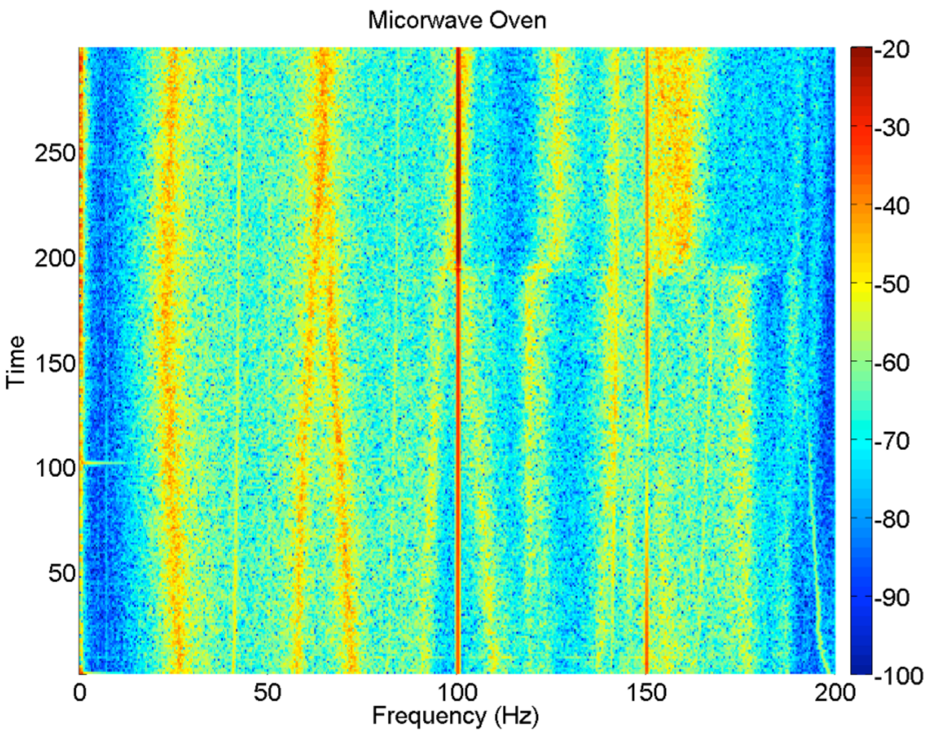


**Accelerometer:**
- Tri axial
- Bandwidth from 0.4Hz to 10kHz
- ±50g

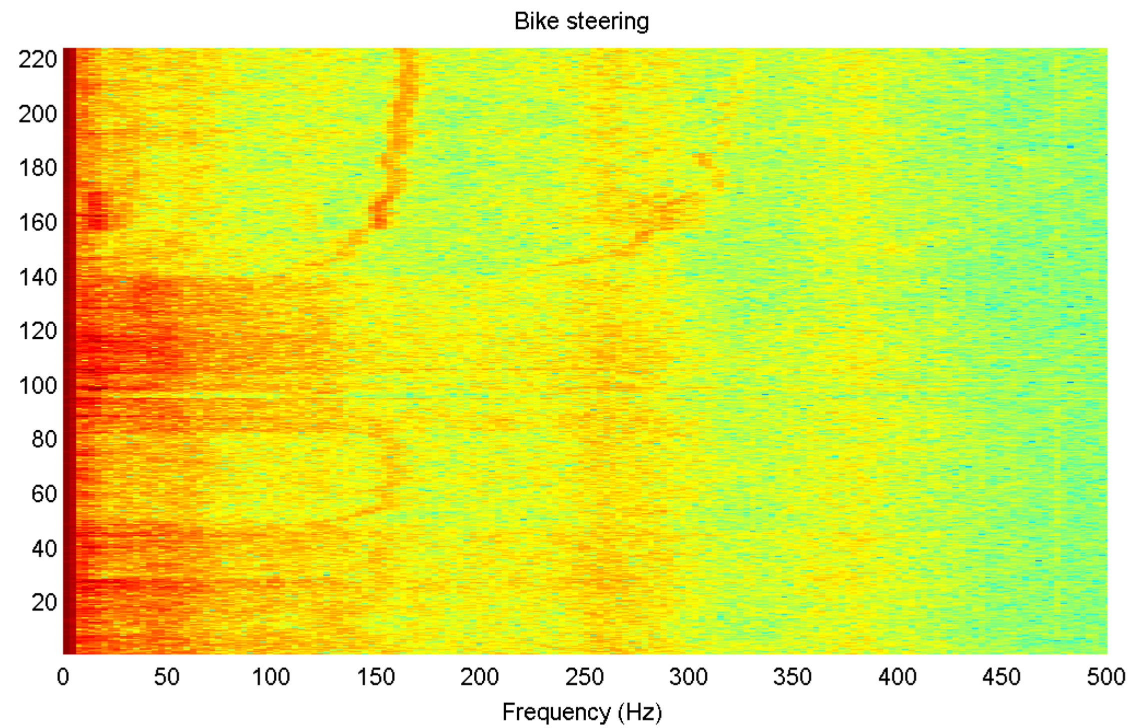
**DAQ:**
- 102.4 kS/s five simultaneous channel
- 4 channels with software-selectable IEPE signal conditioning
- USB powered



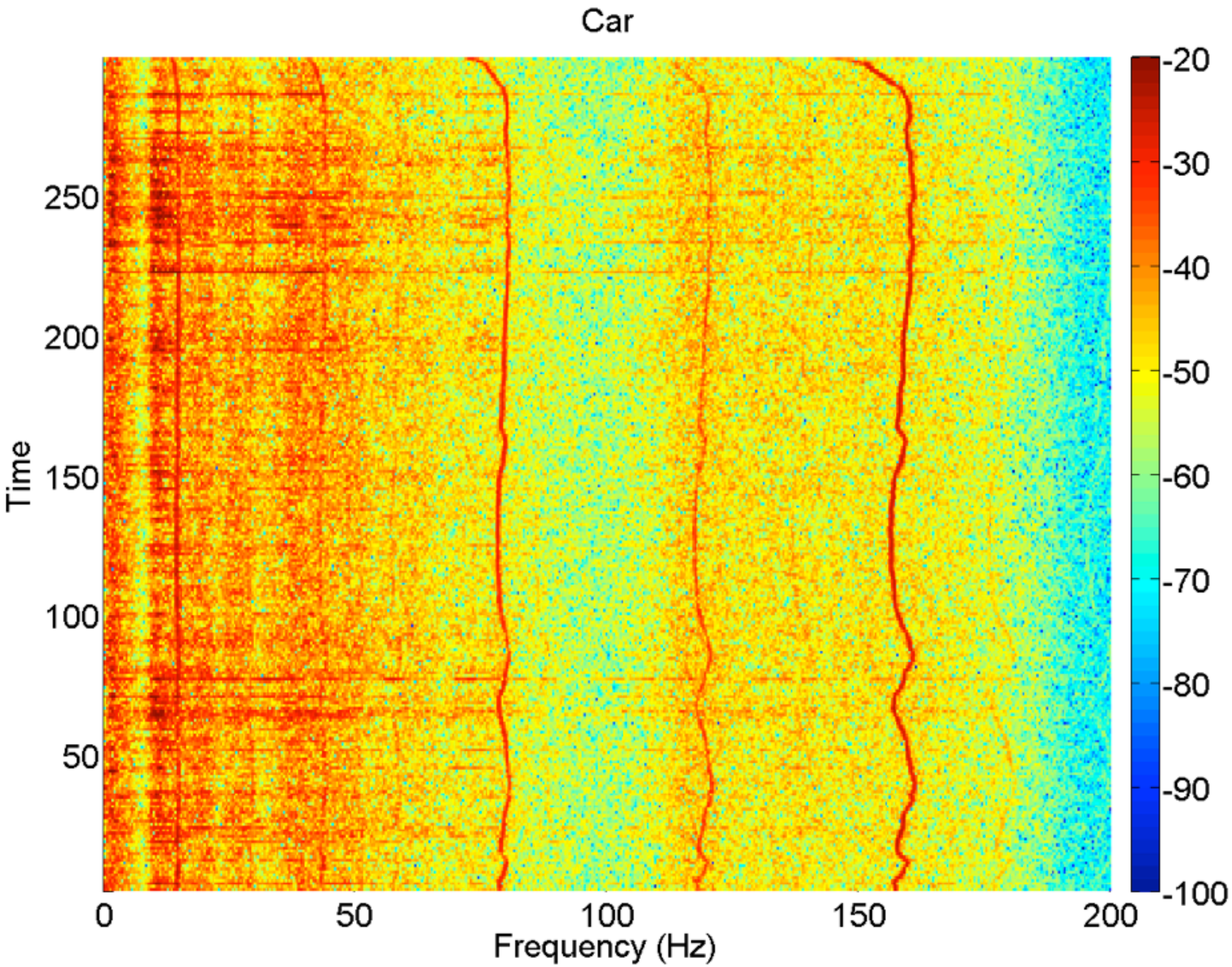

**Signal presentation:**
- Description
- Power spectrum
- Statistical data
- Time series download (only for authorized users)



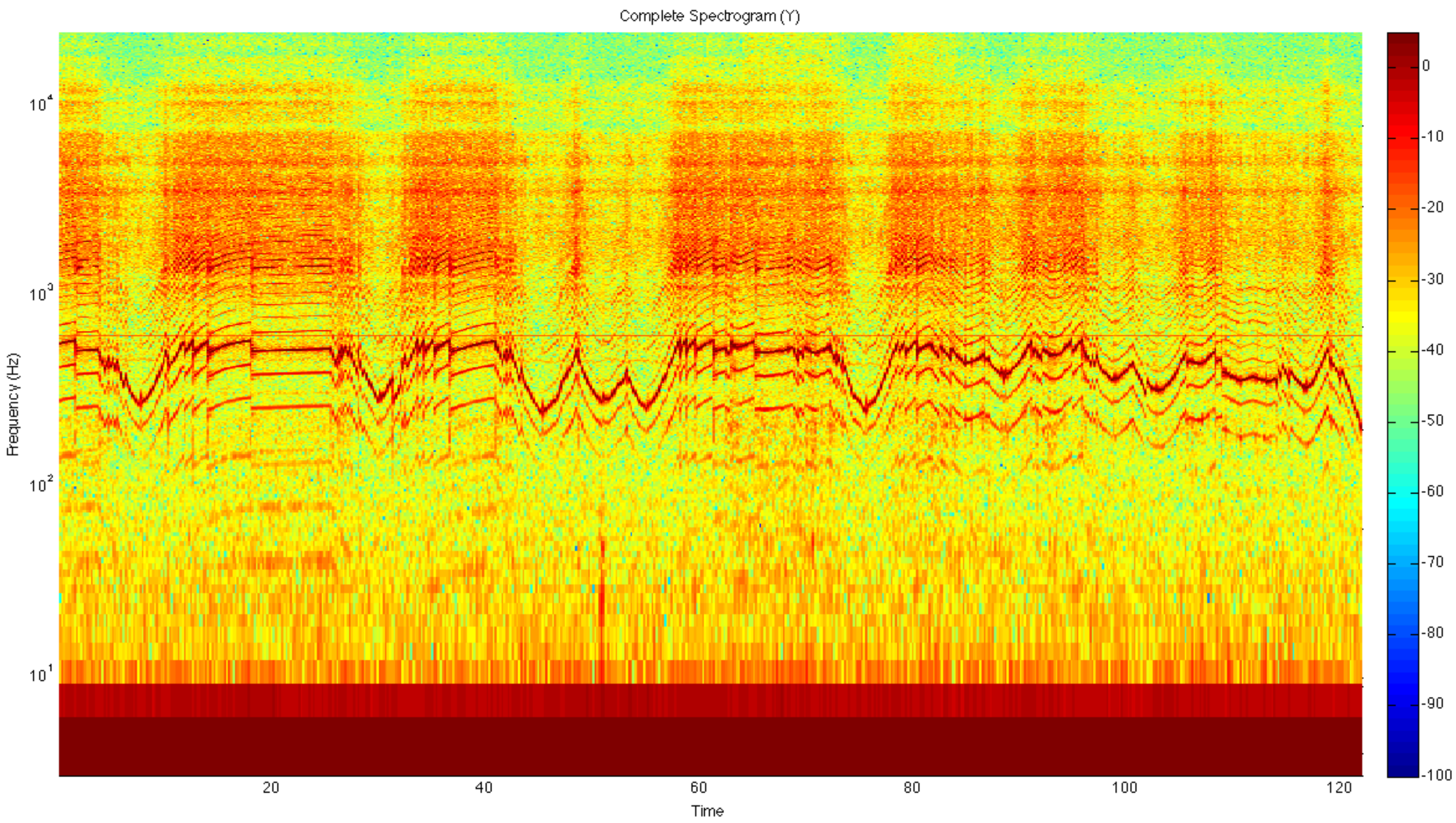
Micorwave Oven
Time
Frequency (Hz)

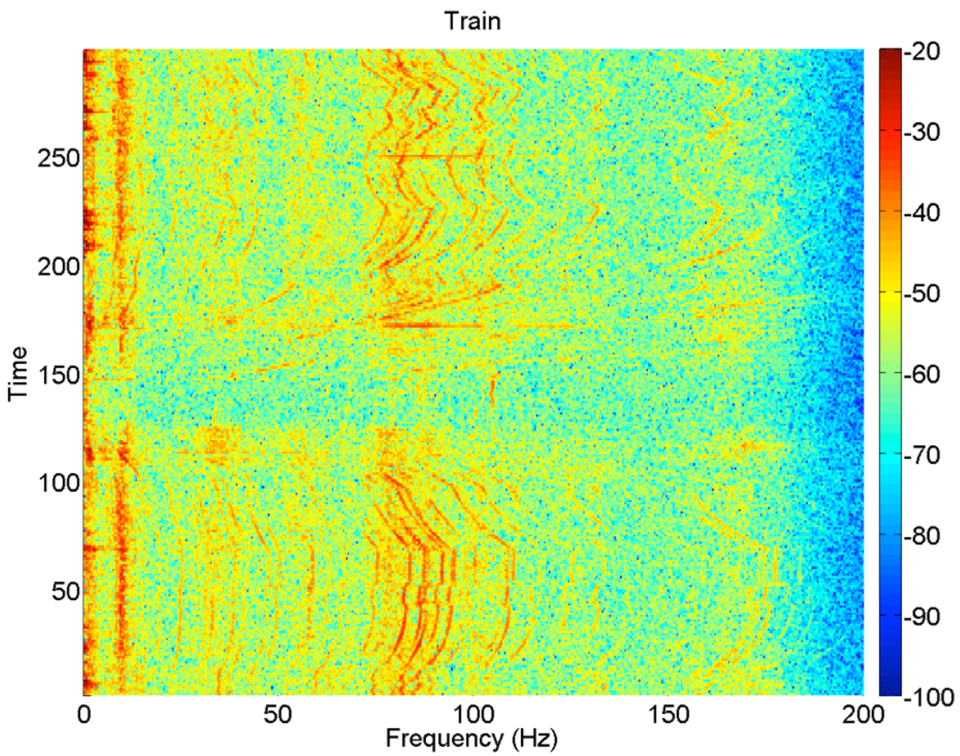
Bike steering
Frequency (Hz)

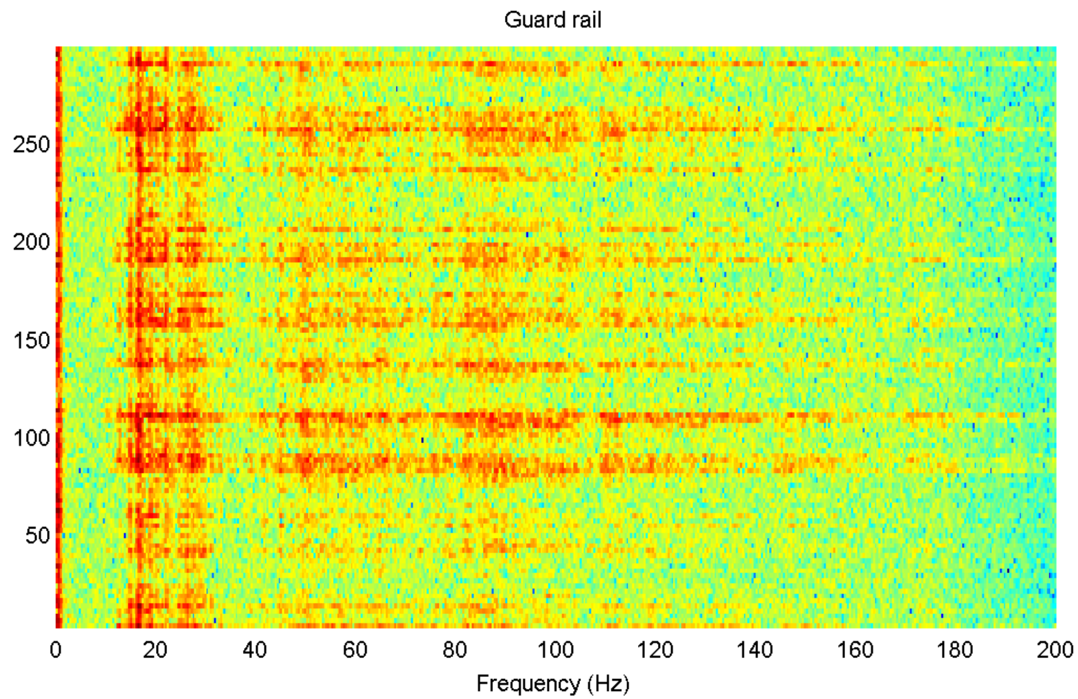

Car
Time
250
200
150
100
50
0
50
100
150
200
Frequency (Hz)
-20
-30
-40
-50
-60
-70
-80
-90
-100

# F1 car

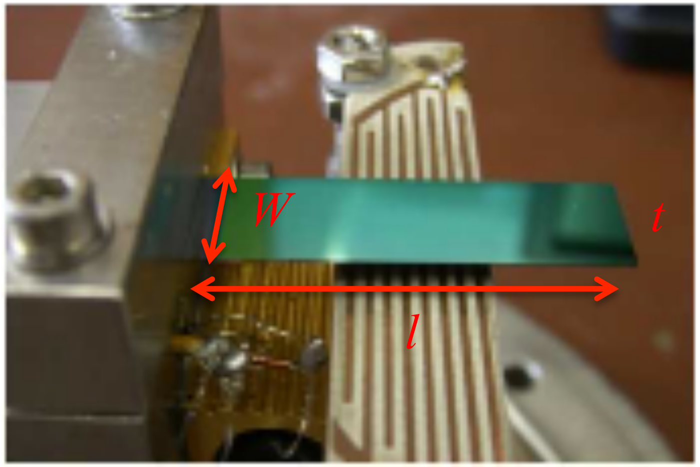


Complete Spectrogram (Y)
Frequency (Hz)
10^4
10^3
10^2
10^1
20
40
60
80
100
120
Time
0
-10
-20
-30
-40
-50
-60
-70
-80
-90
-100


Train
Time
Frequency (Hz)
0
50
100
150
200
250
-20
-30
-40
-50
-60
-70
-80
-90
-100


Guard rail
Frequency (Hz)
0
20
40
60
80
100
120
140
160
180
200
50
150
250

The frequency spectrum of available vibrations is particularly rich in energy in the low frequency part… and it is very difficult, if not impossible, to build small low-frequency resonant systems…




Resonant frequency ~ $[s^{-1}]$

- MEMS cantilever 100 × 3 × 0.1 μm³, $f_0$=12 kHz
- NEMS cantilever 0.1 × 0.01 × 0.01 μm³, $f_0$=1.2 GHz

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \qquad \delta = \frac{Wl^3}{3EI} \qquad k = \frac{W}{\delta} = \frac{3EI}{l^3}$$

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{3EI}{Ml^3}} = \frac{1}{2\pi}\sqrt{\frac{Ewt^3}{4Ml^3}} = \frac{t}{4\pi l^2}\sqrt{\frac{E}{\rho}}$$

# From the model for a linear oscillator:

The voltage transfer function is:

$$|H(\omega)| = \frac{1}{m}\frac{\omega k_c}{\sqrt{\omega^2\left(\frac{k+\frac{\gamma}{\tau}+k_c k_v}{m}-\omega^2\right)^2+\left(\left(\frac{\gamma}{m}+\frac{1}{\tau}\right)\omega^2-\frac{k}{m\tau}\right)^2}}$$

or considering: $\omega_0 = \sqrt{\frac{k+\frac{\gamma}{\tau}+k_c k_v}{m}}$

and: $\omega_1 = \sqrt{\frac{k}{\gamma\tau+m}}$

$$|H(\omega)| = \frac{1}{m}\frac{\omega k_c}{\sqrt{\omega^2\left(\omega^2-\omega_0^2\right)^2+\left(\frac{\gamma\tau+m}{m\tau}\right)^2\left(\omega^2-\omega_1^2\right)^2}}$$

if $\omega^2\left(\omega^2-\omega_0^2\right)^2 > \left(\frac{\gamma\tau+m}{m\tau}\right)^2\left(\omega^2-\omega_1^2\right)^2$

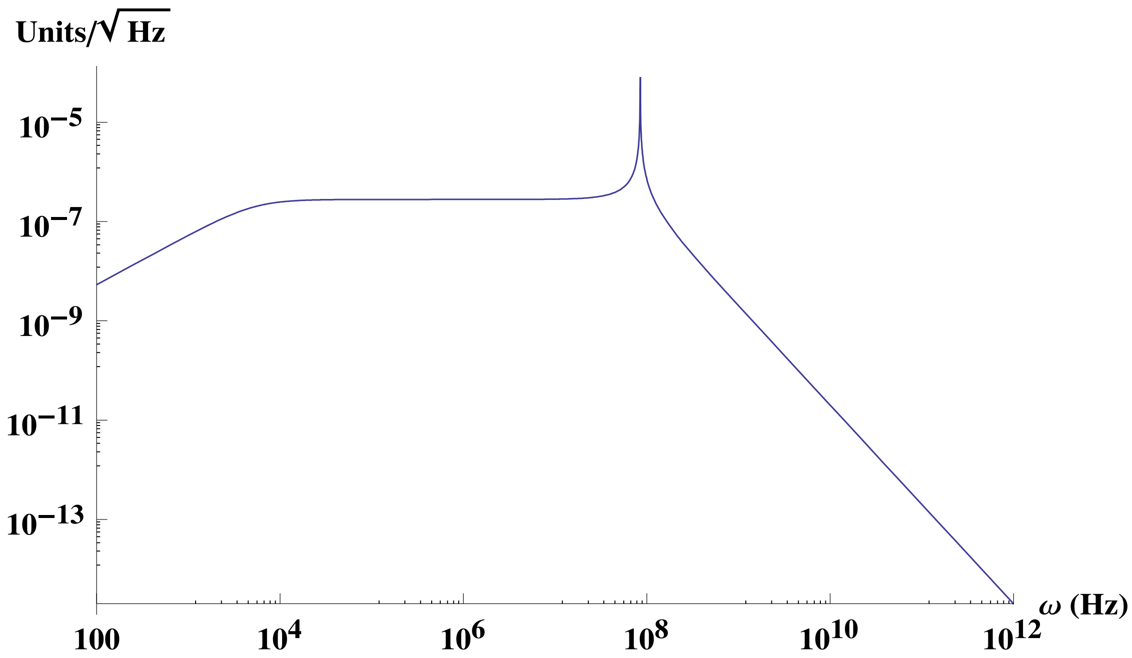
the resonance frequency is $\omega_0 = \sqrt{\frac{k+\frac{\gamma}{\tau}+k_c k_v}{m}}$

where $|H(\omega)|_{\max} = \frac{\omega_0 k_c \tau}{(\gamma\tau+m)\left(\omega_0^2-\omega_1^2\right)^2}$

if $\omega^2\left(\omega^2-\omega_0^2\right)^2 < \left(\frac{\gamma\tau+m}{m\tau}\right)^2\left(\omega^2-\omega_1^2\right)^2$

the resonance frequency is $\omega_1 = \sqrt{\frac{k}{\gamma\tau+m}}$

where $|H(\omega)|_{\max} = \frac{k_c}{m\left|\omega_1^2-\omega_0^2\right|}$



# The analytic result for the Q

$$Q = \frac{\omega_r}{\Delta\omega}$$

$\omega_r$ is the resonance frequency and $\Delta\omega$ is the bandwidth (full width when the output voltage is $V_{\max}/\sqrt{2}$ )

$$
\begin{aligned}
\text{Quality Factor} = \Bigg\{ \Bigg( & 3\, k\, m\, \tau^2 \sqrt{\Big(2 m^2 + 2 (k + kc\, kv)\, m\, \tau^2 - \gamma^2 \tau^2 +} \\
& \sqrt{m^4 + 2 (k - 2\, kc\, kv)\, m^3 \tau^2 - 4 (k + kc\, kv)\, m\, \gamma^2 \tau^4 + \gamma^4 \tau^4 + m^2 \tau^2 \left(-\gamma^2 - 6\, kc\, kv\, \gamma\, \tau + (k + kc\, kv)^2 \tau^2\right)}\Big) \Bigg) \Bigg/ \Bigg( \sqrt{\Big(2 m^6 + 6 (k} \\
& - 2\, kc\, kv)\, m^5 \tau^2 + 2 \gamma^6 \tau^6 - 2 \gamma^4 \tau^4 \\
& \sqrt{m^4 + 2 (k - 2\, kc\, kv)\, m^3 \tau^2 - 4 (k + kc\, kv)\, m\, \gamma^2 \tau^4 + \gamma^4 \tau^4 + m^2 \tau^2 \left(-\gamma^2 - 6\, kc\, kv\, \gamma\, \tau + (k + kc\, kv)^2 \tau^2\right)} + 4 (k + kc\, kv)\, m\, \gamma^2 \\
& \tau^4 \left(-3 \gamma^2 \tau^2 + 2 \sqrt{m^4 + 2 (k - 2\, kc\, kv)\, m^3 \tau^2 - 4 (k + kc\, kv)\, m\, \gamma^2 \tau^4 + \gamma^4 \tau^4 + m^2 \tau^2 \left(-\gamma^2 - 6\, kc\, kv\, \gamma\, \tau + (k + kc\, kv)^2 \tau^2\right)}\right) - \\
& m^4 \Big(3 \gamma^2 \tau^2 + 18\, kc\, kv\, \gamma\, \tau^3 - 6 k^2 \tau^4 + 6\, k\, kc\, kv\, \tau^4 - 15\, kc^2\, kv^2 \tau^4 + 2 \\
& \sqrt{m^4 + 2 (k - 2\, kc\, kv)\, m^3 \tau^2 - 4 (k + kc\, kv)\, m\, \gamma^2 \tau^4 + \gamma^4 \tau^4 + m^2 \tau^2 \left(-\gamma^2 - 6\, kc\, kv\, \gamma\, \tau + (k + kc\, kv)^2 \tau^2\right)}\Big) + m^2 \tau^2 \Big(-3 \gamma^4 \\
& \tau^2 - 18\, kc\, kv\, \gamma^3 \tau^3 + 12\, kc\, kv\, \gamma\, \tau \\
& \sqrt{m^4 + 2 (k - 2\, kc\, kv)\, m^3 \tau^2 - 4 (k + kc\, kv)\, m\, \gamma^2 \tau^4 + \gamma^4 \tau^4 + m^2 \tau^2 \left(-\gamma^2 - 6\, kc\, kv\, \gamma\, \tau + (k + kc\, kv)^2 \tau^2\right)} - 2 (k + kc\, kv)^2 \\
& \tau^2 \sqrt{m^4 + 2 (k - 2\, kc\, kv)\, m^3 \tau^2 - 4 (k + kc\, kv)\, m\, \gamma^2 \tau^4 + \gamma^4 \tau^4 + m^2 \tau^2 \left(-\gamma^2 - 6\, kc\, kv\, \gamma\, \tau + (k + kc\, kv)^2 \tau^2\right)} + \gamma^2 \Big(15 k^2 \tau^4 \\
& + 30\, k\, kc\, kv\, \tau^4 + 15\, kc^2\, kv^2 \tau^4 + 2 \\
& \sqrt{m^4 + 2 (k - 2\, kc\, kv)\, m^3 \tau^2 - 4 (k + kc\, kv)\, m\, \gamma^2 \tau^4 + \gamma^4 \tau^4 + m^2 \tau^2 \left(-\gamma^2 - 6\, kc\, kv\, \gamma\, \tau + (k + kc\, kv)^2 \tau^2\right)}\Big)\Big) + 2 m^3 \tau^2 \Big( \\
& k^3 \tau^4 + 3 k^2\, kc\, kv\, \tau^4 + k \Big(6 \gamma^2 \tau^2 + 18\, kc\, kv\, \gamma\, \tau^3 + 3\, kc^2\, kv^2 \tau^4 - 2 \\
& \sqrt{m^4 + 2 (k - 2\, kc\, kv)\, m^3 \tau^2 - 4 (k + kc\, kv)\, m\, \gamma^2 \tau^4 + \gamma^4 \tau^4 + m^2 \tau^2 \left(-\gamma^2 - 6\, kc\, kv\, \gamma\, \tau + (k + kc\, kv)^2 \tau^2\right)}\Big) + kc\, kv \Big(-3 \\
& \gamma^2 \tau^2 + 18\, kc\, kv\, \gamma\, \tau^3 + kc^2\, kv^2 \tau^4 + 4 \\
& \sqrt{m^4 + 2 (k - 2\, kc\, kv)\, m^3 \tau^2 - 4 (k + kc\, kv)\, m\, \gamma^2 \tau^4 + \gamma^4 \tau^4 + m^2 \tau^2 \left(-\gamma^2 - 6\, kc\, kv\, \gamma\, \tau + (k + kc\, kv)^2 \tau^2\right)}\Big)\Big)\Big)} \Bigg) \Bigg\}
\end{aligned}
$$

## Description of the resonator design

The resonator design is a square shaped block of single crystal silicon with dimensions of 320x320x28 um^3 (design H1). Its main resonance mode is the so called square extensional (SE) resonance, which is characterized by its zoom-in/zoom-out oscillation. The resonance is excited by a piezoelectric AlN thin film on top of the resonator block. The electrically conductive (p-doped) silicon block acts as the bottom electrode, and a molybdenum thin film has been patterned to provide the top electrode. See reference [1] for a general description of the SE resonator. Reference [2] discusses piezoelectric excitation of the SE resonance mode.

Figure 3 shows how the resonator is recommended to be connected.

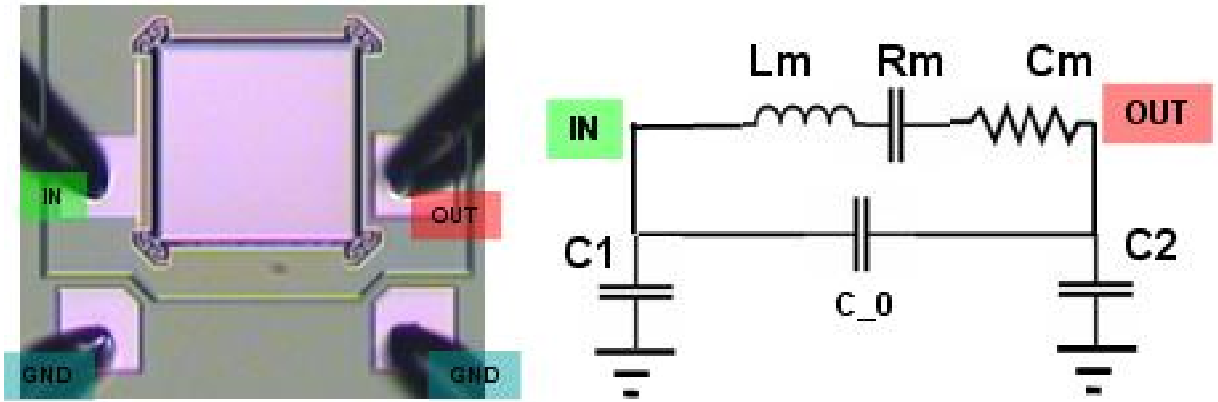
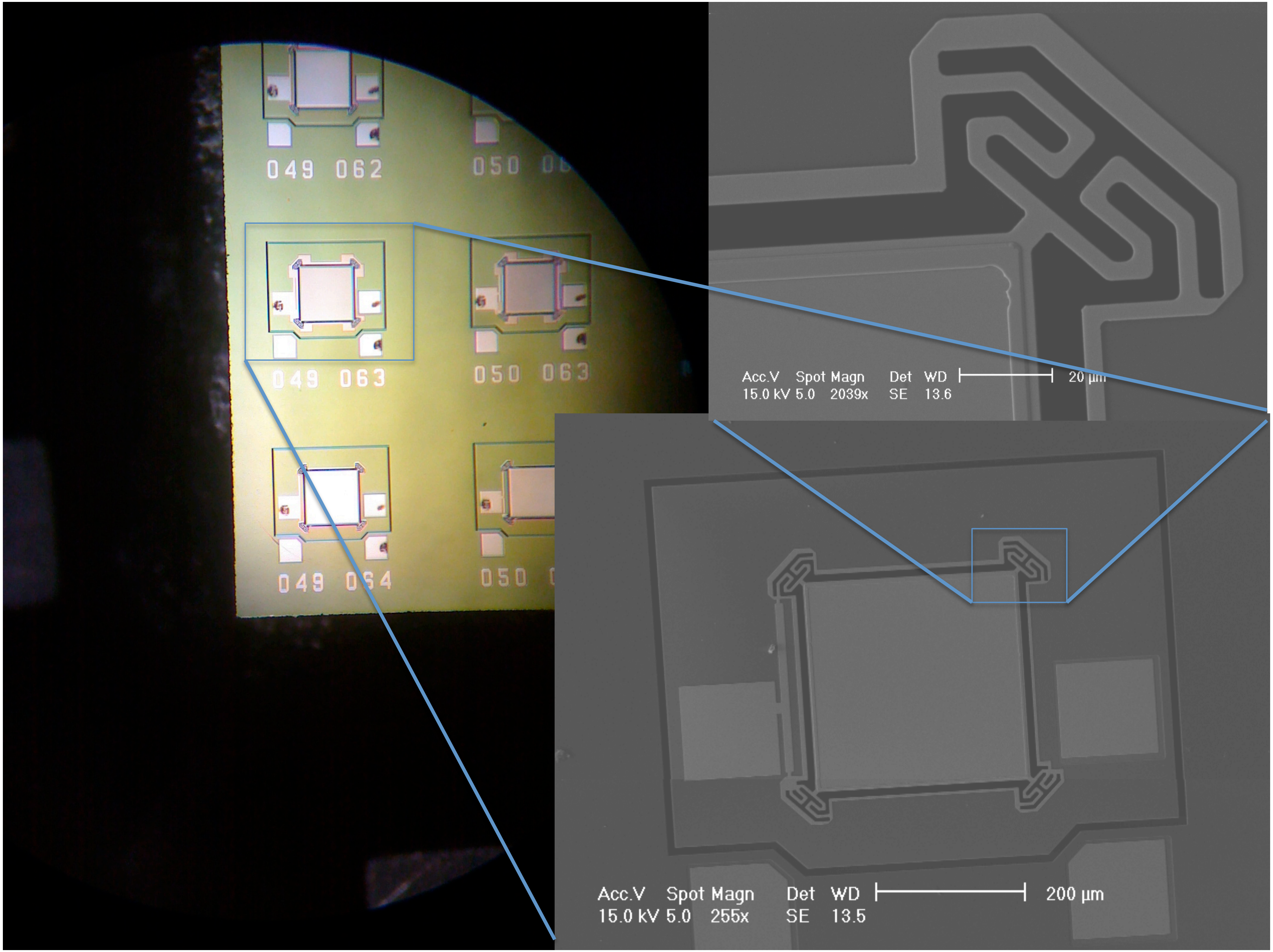



*Figure 3: Electrical connection of the resonator.*


References

[1] V. Kaajakari et al., "Square-Extensional Mode Single-Crystal Silicon Micromechanical Resonator for Low-Phase-Noise Oscillator Applications," *IEEE Electron Device Letters* 25, no. 4 (4, 2004): 173-175.

[2] A. Jaakkola et al., "Piezoelectrically transduced Single-Crystal-Silicon Plate Resonators," in *IEEE Ultrasonics Symposium* (presented at the IEEE Ultrasonics Symposium, Beijing, China, 2008), 2181 – 2184


049 062
050
049 063
050 063
049 064
050
Acc.V Spot Magn Det WD 20 µm
15.0 kV 5.0 2039x SE 13.6
Acc.V Spot Magn Det WD 200 µm
15.0 kV 5.0 255x SE 13.5

# The measurements

**Resonators design**

Four separate chips have been provided, each chip contains 16 resonators.

Each of the 16 resonators has a different design, their size is varied (so the resonance frequency)

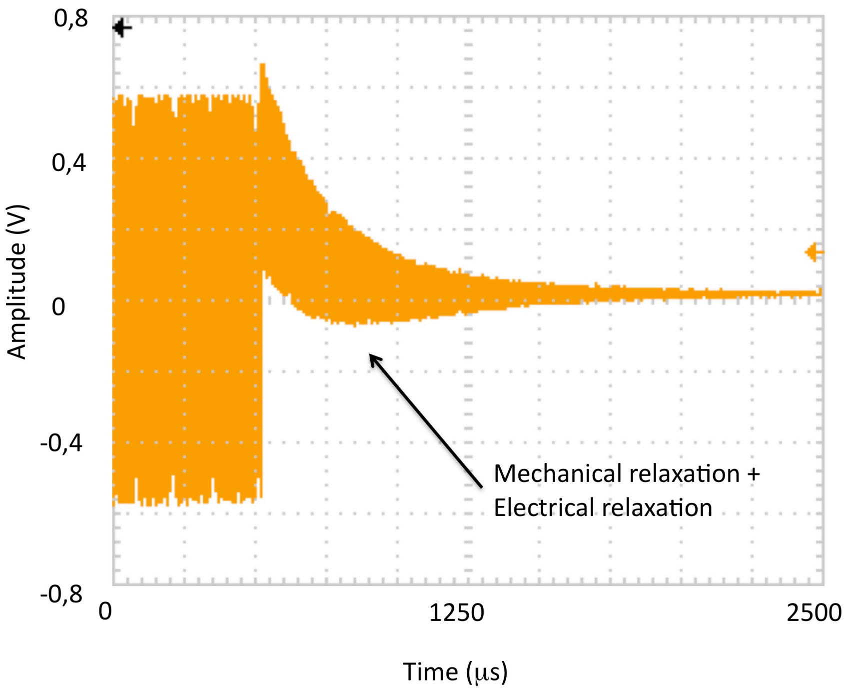
**resonator lateral size (um)**

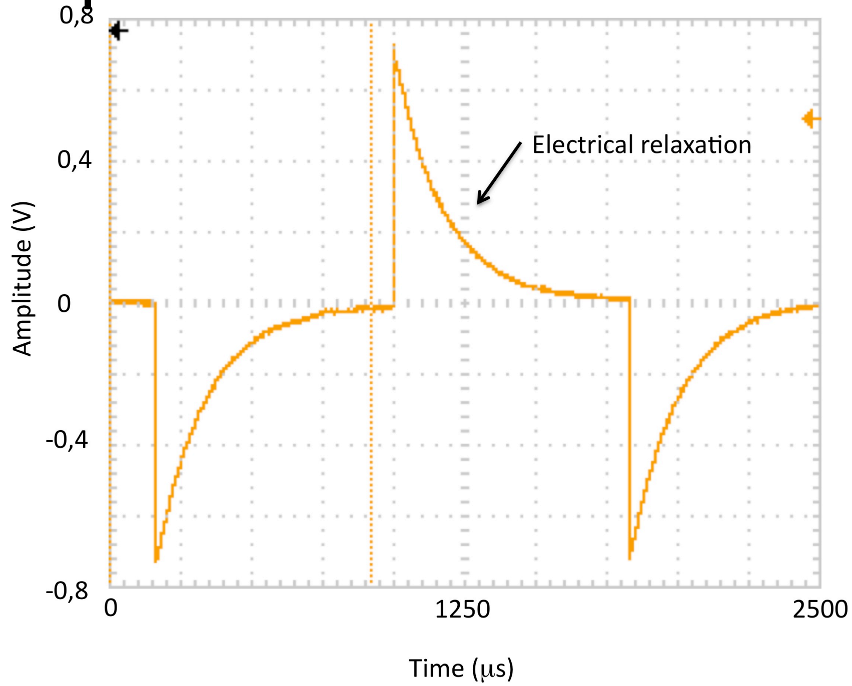
| | | | |
|---|---|---|---|
| 311 | 287 | 263 | 239 |
| 314 | 290 | 266 | 242 |
| 317 | 293 | 269 | 245 |
| 320 | 296 | 272 | 248 |

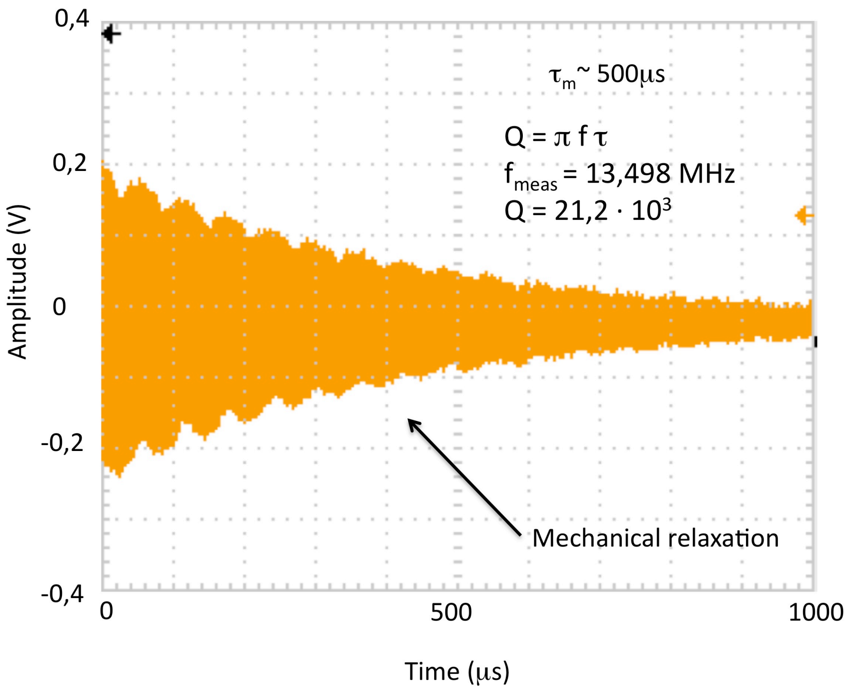
**main mode frquency (MHz)**

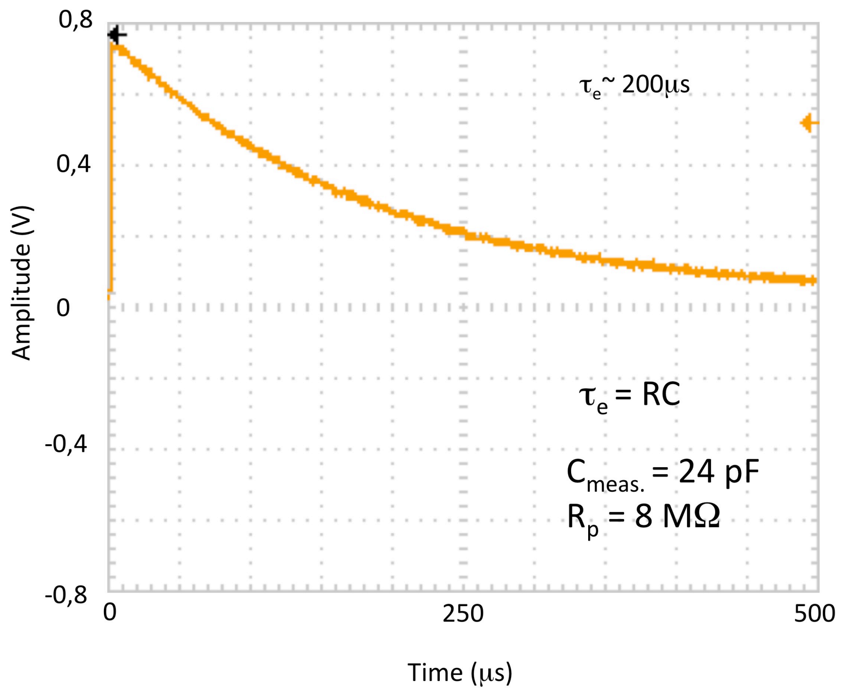
| | | | |
|---|---|---|---|
| 13.9 | 15.1 | 16.4 | 18.1 |
| 13.8 | 14.9 | 16.2 | 17.9 |
| 13.6 | 14.7 | 16.1 | 17.6 |
| 13.5 | 14.6 | 15.9 | 17.4 |

# Membrane response



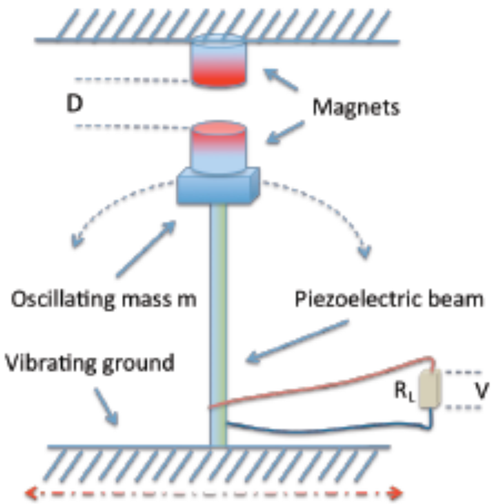



# The vibration harvester 2.0

- ✓ Capable of harvesting energy on a broad-band ------------> Non-resonant system
- ✓ No need for frequency tuning ------------> “Transfer function” with wide frequency resp.
- ✓ Capable of harvesting energy at low frequency ------------> Low frequency operated

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**Nonlinear Energy Harvesting**

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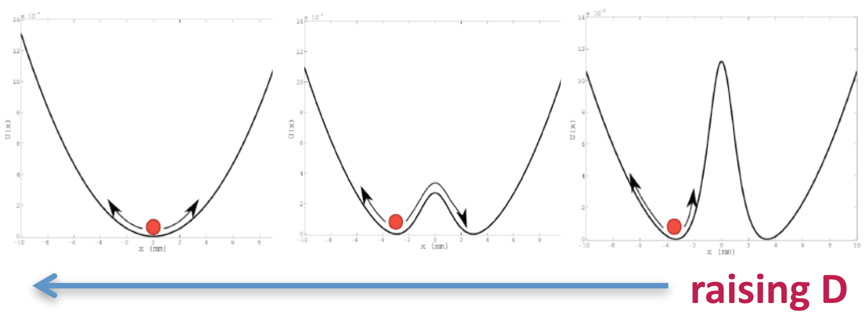
Ambient energy harvesting has been in recent years the recurring object of a number of research efforts aimed at providing an autonomous solution to the powering of small-scale electronic mobile devices. Among the different solutions, vibration energy harvesting has played a major role due to the almost universal presence of mechanical vibrations. Here we propose a new method based on the exploitation of the dynamical features of stochastic nonlinear oscillators. Such a method is shown to outperform standard linear oscillators and to overcome some of the most severe limitations of present approaches. We demonstrate the superior performances of this method by applying it to piezoelectric energy harvesting from ambient vibration.

DOI: 10.1103/PhysRevLett.102.080601 PACS numbers: 05.40.Ca, 05.10.Ln, 05.45.−a, 84.60.−h




Result: output power is maximum for an optimal nonlinear regime

Let's look at an example of non-linear oscillator:

the Duffing Oscillator

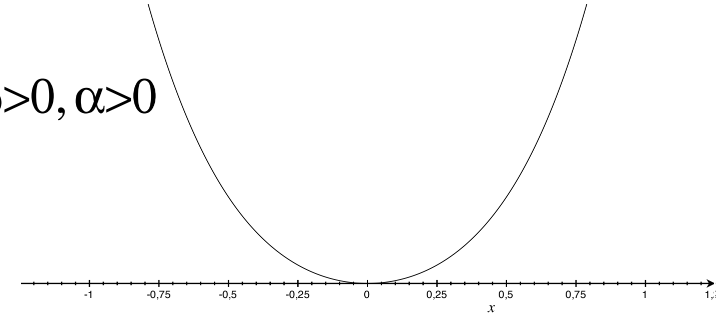
$$\ddot{x} + \delta\dot{x} + \beta x + \alpha x^3 = \gamma \cos \omega t$$

$$U(x) = \frac{1}{2}\beta x^2 + \frac{1}{4}\alpha x^4$$

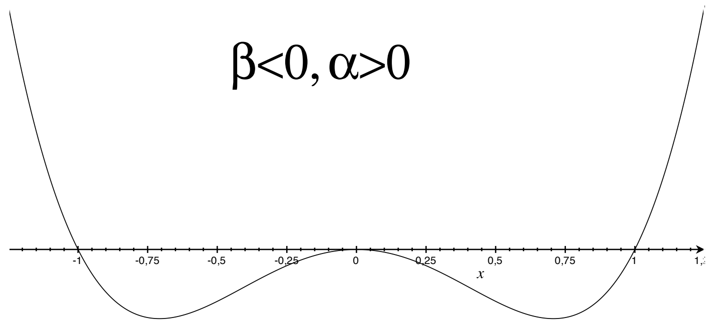
# The Duffing Potential

$$U(x) = \frac{1}{2}\beta x^2 + \frac{1}{4}\alpha x^4$$

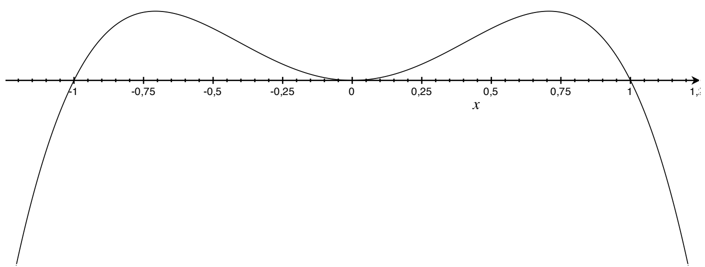
$\beta > 0, \alpha > 0$



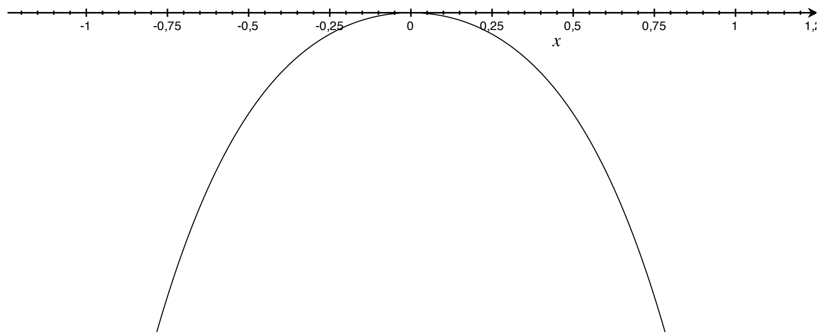

$\beta < 0, \alpha > 0$



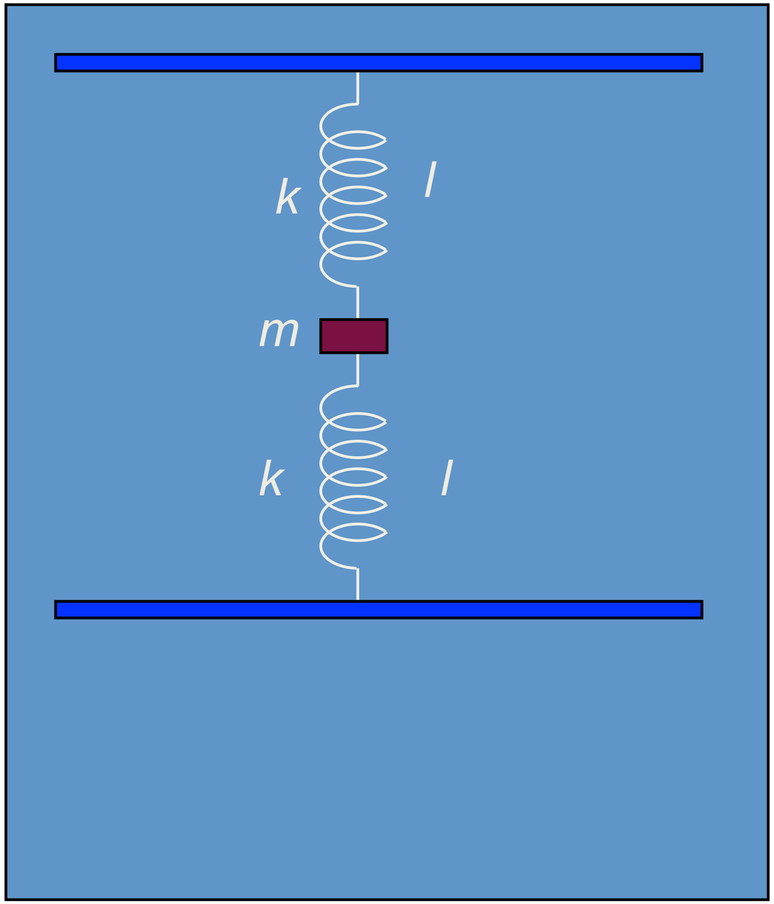

$\beta > 0, \alpha < 0$




$\beta < 0, \alpha < 0$



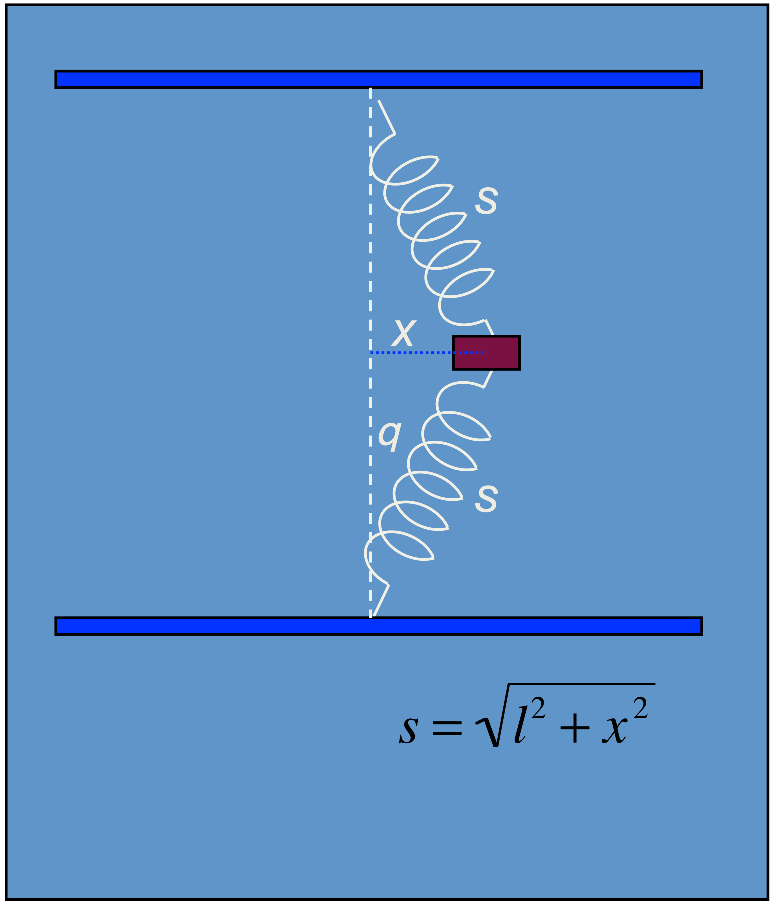
# The example of a two springs system




- A mass is held between two springs.
  - Spring constant *k*
  - Natural length *l*

- Springs are on a horizontal surface.
  - Frictionless
  - No gravity

# Transverse Displacement




- The force for a displacement is due to both springs.
  - Only transverse component
  - Looks like its harmonic

$$F = -2k\left(\sqrt{l^2 + x^2} - l\right)\sin\theta$$

$$= -2k\left(\sqrt{l^2 + x^2} - l\right)\frac{x}{\sqrt{l^2 + x^2}}$$

$$= -2kx\left(1 - \frac{1}{\sqrt{1 + x^2/l^2}}\right)$$

# Purely Nonlinear

The force can be expanded as a power series near equilibrium (in 0).

- Expand in *x/l*

$$F = -2kl\frac{x}{l}\left(1 - \frac{1}{\sqrt{1 + x^2/l^2}}\right)$$

The lowest order term is non-linear.

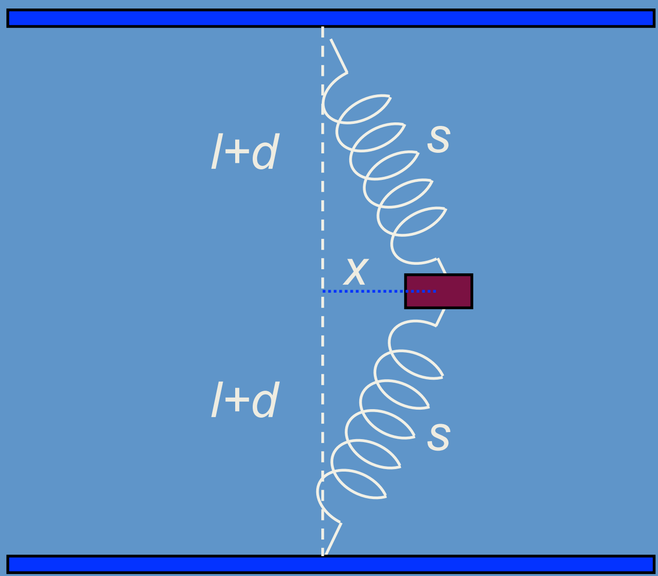
$$F \cong -kl\left(\frac{x}{l}\right)^3 + \ldots$$

Quartic potential

- Not just a perturbation

$$V \cong \frac{k}{4l^2}x^4 + \ldots$$

# Mixed Potential




with $d < l$

Typical springs are not at natural length $l$

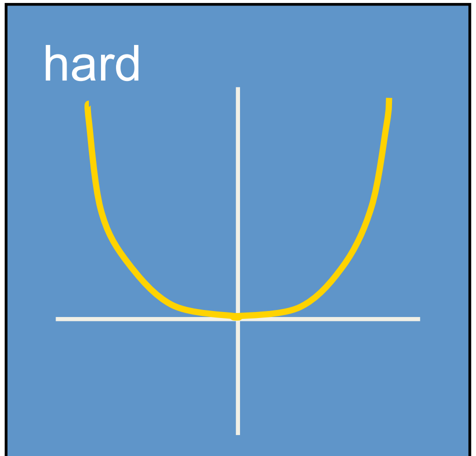
- Approximation includes a linear term

$$F \cong -\frac{2kd}{l}x - \frac{k(l-d)}{l^3}x^3 + \ldots$$

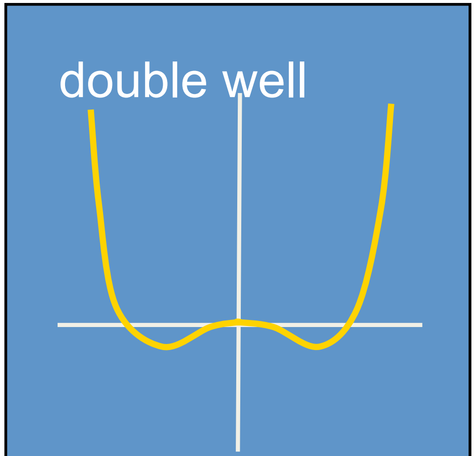
$$V \cong \frac{kd}{l}x^2 + \frac{k(l-d)}{4l^3}x^4 + \ldots$$

# Quartic Potentials

- The sign of the forces influence the shape of the potential.




$$V = \frac{k}{2}x^2 + \frac{k\alpha}{4}x^4$$




$$V = -\frac{k}{2}x^2 + \frac{k\alpha}{4}x^4$$

# Driven System (for the hardening case)

Assuming a more complete, realistic system.

$$m\ddot{x} = -\beta\dot{x} - kx - k\alpha x^3 + f\cos\omega t$$

- Damping term
- Driving force

We rescale the problem:

- Set $t$ such that $\omega_o^2 = (k/m) = 1$
- Set $x$ such that $k\alpha/m = 1$

$$\ddot{x} + \gamma\dot{x} + \omega_0^2 x + \alpha\omega_0^2 x^3 = f\cos\omega t$$

We obtain the classical Duffing equation

$$\ddot{x} + \gamma\dot{x} + x + x^3 = f\cos\omega t$$

# Steady State Solution

## Try a solution, match terms

$x(t) = A(\omega)\cos[\omega t - \theta]$ in $\ddot{x} + \gamma\dot{x} + x + x^3 = f\cos\omega t$

$$A(1-\omega^2)\cos(\omega t - \theta) - A\gamma\omega\sin(\omega t - \theta) + A^3\cos^3(\omega t - \theta) = f\cos\omega t$$

trigonometric identities

$$\cos^3(\omega t - \theta) = \tfrac{3}{4}\cos(\omega t - \theta) + \tfrac{1}{4}\cos 3(\omega t - \theta)$$

$$f\cos\omega t = f\cos\theta\cos(\omega t - \theta) - f\sin\theta\sin(\omega t - \theta)$$

$$[A(1-\omega^2 + \tfrac{3}{4}A^2) - f\cos\omega t]\cos(\omega t - \theta) \rightarrow f\cos\omega t = A(1-\omega^2+\tfrac{3}{4}A^2)$$

$$+[-A\gamma\omega + f\sin\omega t]\sin(\omega t - \theta) \rightarrow f\sin\omega t = A\gamma\omega$$

$$+\tfrac{1}{4}A^3\cos 3(\omega t - \theta) \rightarrow \tfrac{1}{4}A^3\cos 3(\omega t - \theta) \approx 0$$

$$= 0$$

# Amplitude Dependence

$$f^2 \cos^2 \omega t = A^2 (1-\omega^2 + \tfrac{3}{4} A^2)^2$$
$$f^2 \sin^2 \omega t = A^2 \gamma^2 \omega^2$$

$$\Rightarrow \quad f^2 = A^2[(1-\omega^2 + \tfrac{3}{4} A^2)^2 + \gamma^2 \omega^2]$$

Finding the amplitude-frequency relationship:

*Which reduces to forced harmonic oscillator for A ≈ 0*

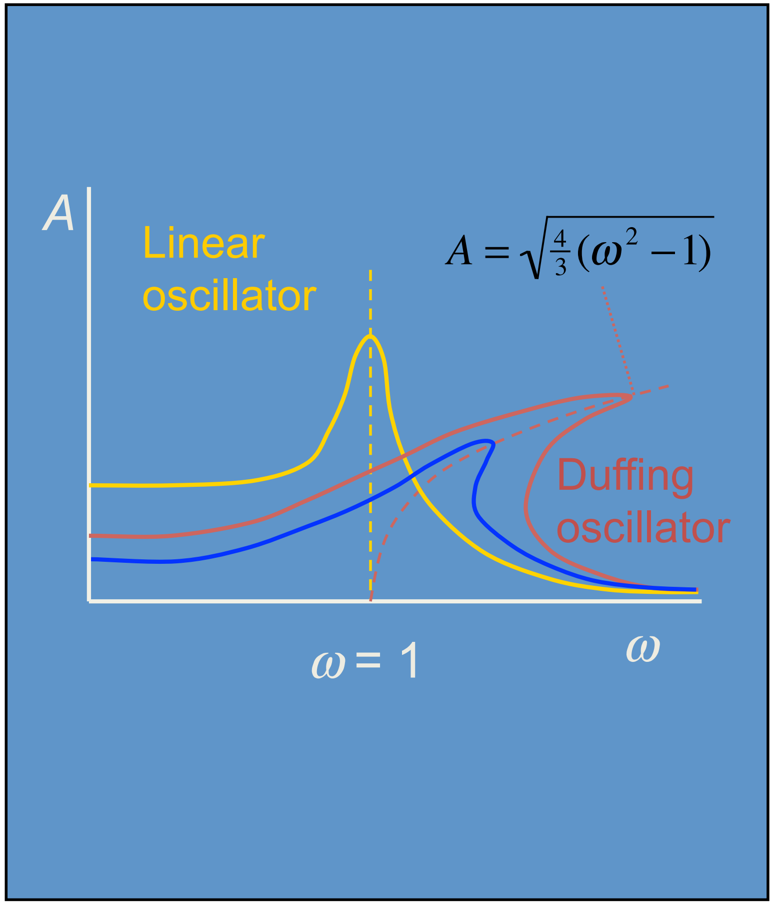
$$A = \frac{f}{\sqrt{[(1-\omega^2)^2 + (\gamma\omega)^2]}}$$

*The harmonic resonance condition is when:* $1-\omega^2 = 0$

In general is: $(1-\omega^2 + \tfrac{3}{4} A^2)^2 = 0$

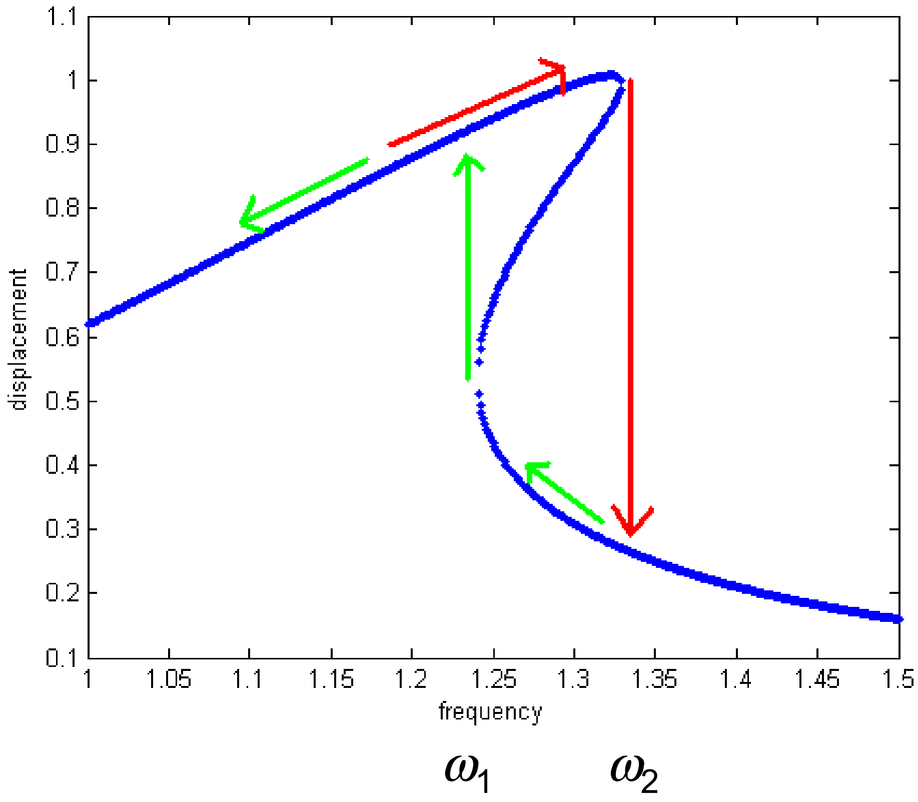
where is valid the relation: $A(\omega) = \sqrt{\tfrac{4}{3}(\omega^2 - 1)}$

# Nonlinear Resonance Frequency




- The resonance frequency of a linear oscillator is independent of amplitude.
- The resonance frequency of a Duffing oscillator increases with amplitude.

# … brings to hysteresis




- A Duffing oscillator behaves differently for increasing and decreasing frequencies.
  - Increasing frequency has a jump in amplitude at $\omega_2$
  - Decreasing frequency has a jump in amplitude at $\omega_1$

- This is hysteresis.

# Nonlinear Resonance

(in general…)

Nonlinear resonance seems not to be so much different from the (linear) resonance of a harmonic oscillator. But:

- the dependency of the eigenfrequency of a nonlinear oscillator on the amplitude

and

- the nonharmoniticity of the oscillation

lead to a behavior that is impossible in harmonic oscillators, namely: the foldover effect and superharmonic resonance.

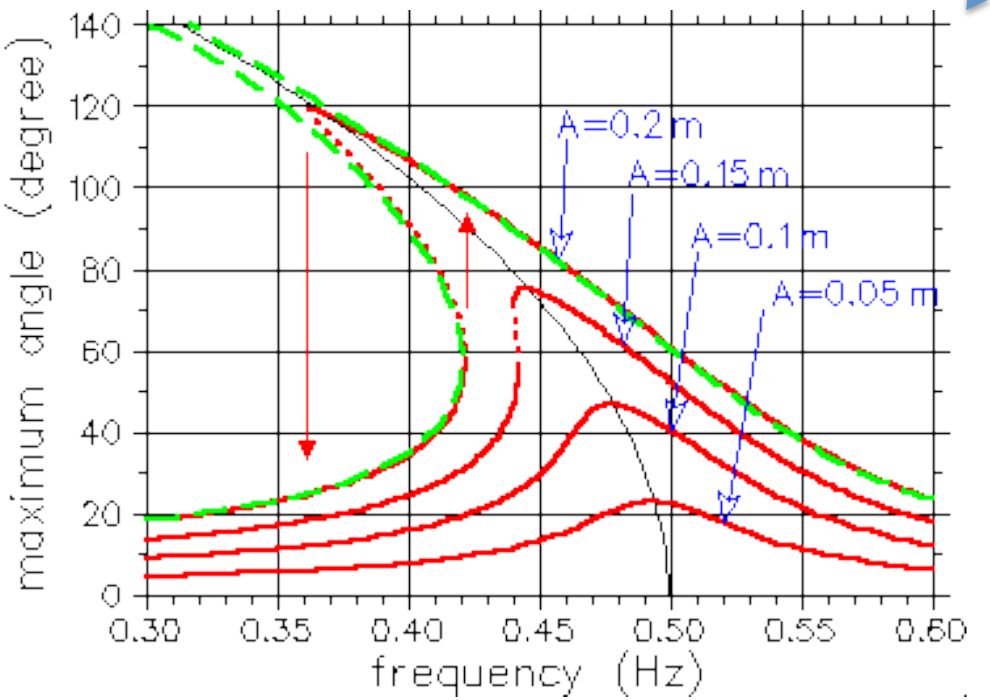
*Both effects are more important in the case of weak damping.*

# The foldover effect

The foldover effect got its name from the bending of the resonance peak in a amplitude versus frequency plot. This bending is due to the frequency-amplitude relation which is typical for nonlinear oscillators.

Foldover effect for a pendulum

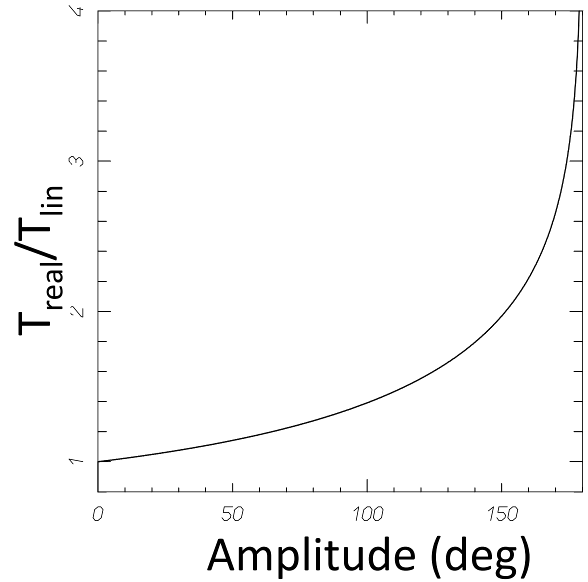
$$g = 9.81 m/\sec^2, l = 1m, \gamma = 0.4 \sec^{-1}$$




The pendulum eq.:

$$\ddot{\varphi} = -\gamma\dot{\varphi} - \omega_0^2 \sin\varphi + f\cos\omega t$$

$$\omega_0^2 = \frac{g}{l}$$

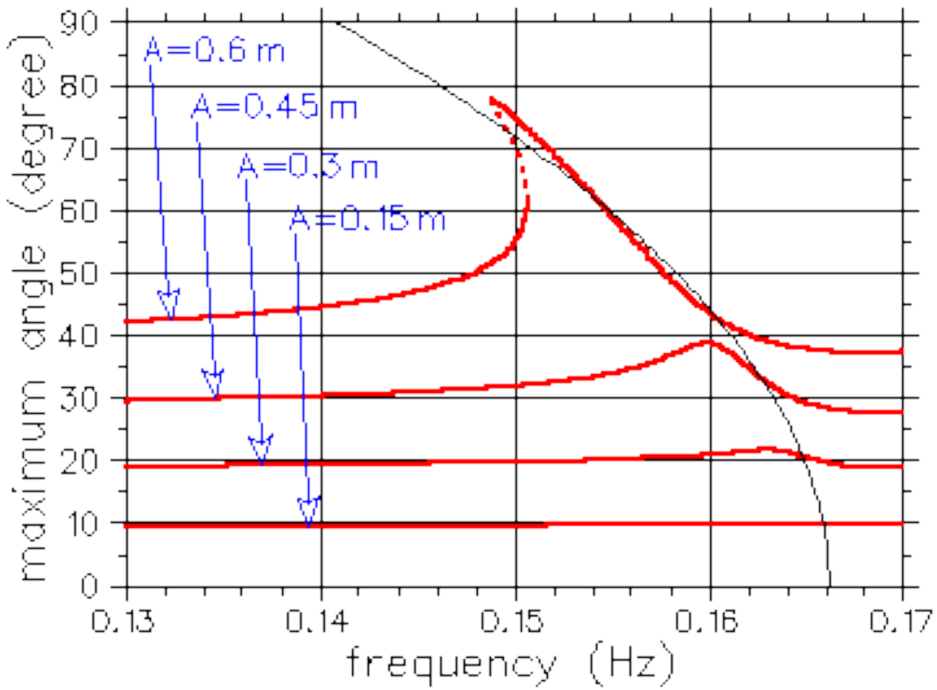


# The superharmonic resonance

***Nonlinear oscillators do not oscillate sinusoidal.***

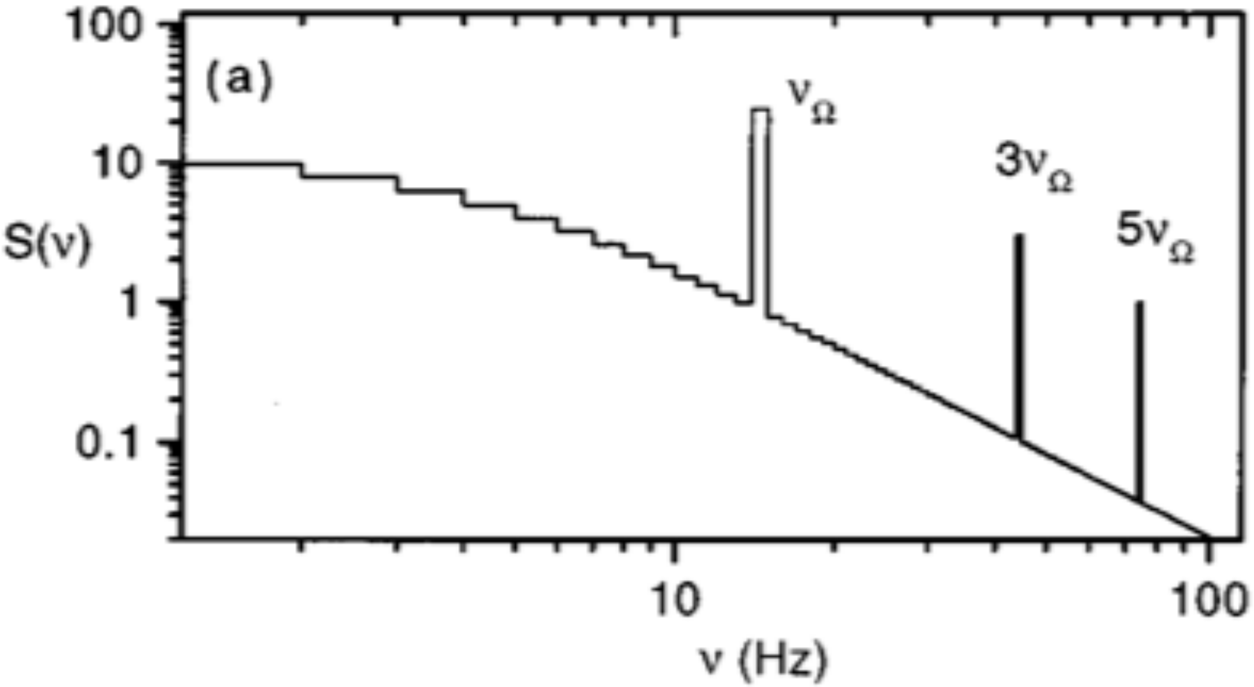
Superharmonic resonance is simply the resonance with one of this higher harmonics of a nonlinear oscillation. In an amplitude/frequency plot appear additional resonance peaks. In general, they appear at driving frequencies which are integer fractions of the fundamental frequency.

$$g = 9.81 m/\sec^2, l = 1m, \gamma = 0.1 \sec^{-1}$$



# Bistable Duffing

In case of a bistable oscillator the frequency response for an overdamped system is highly spread in the low frequency region.

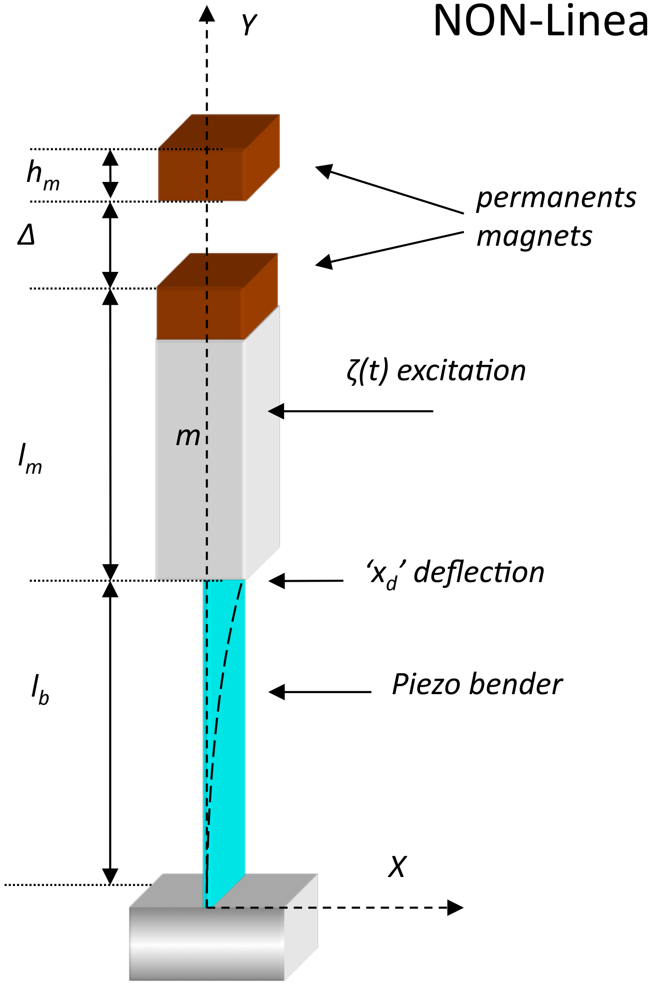


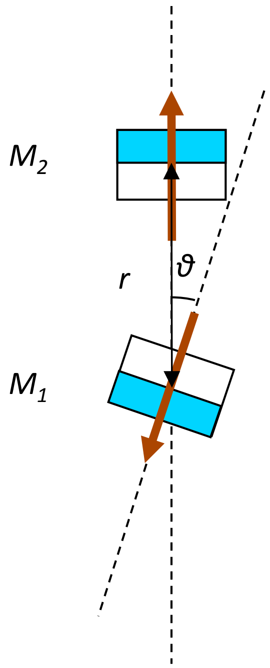

Gammaitoni et al. Reviews of Modern Physics 1998

# Noise energy harvesting

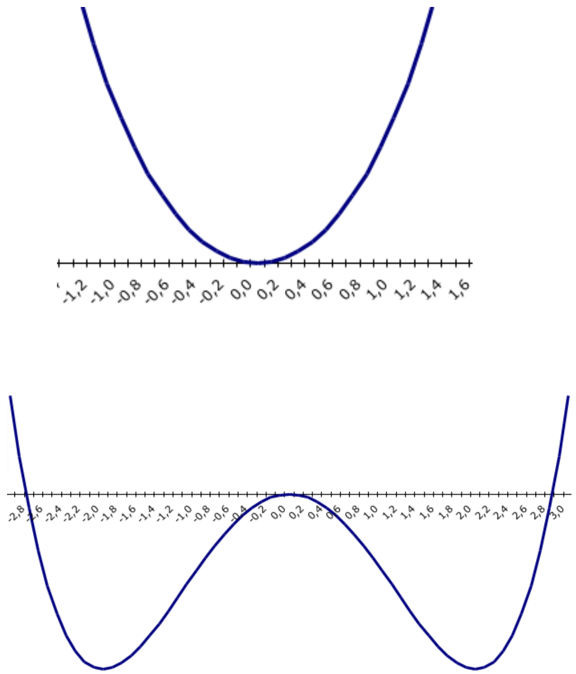
NON-Linear mechanical oscillators

NON-Linear Inverted pendulum



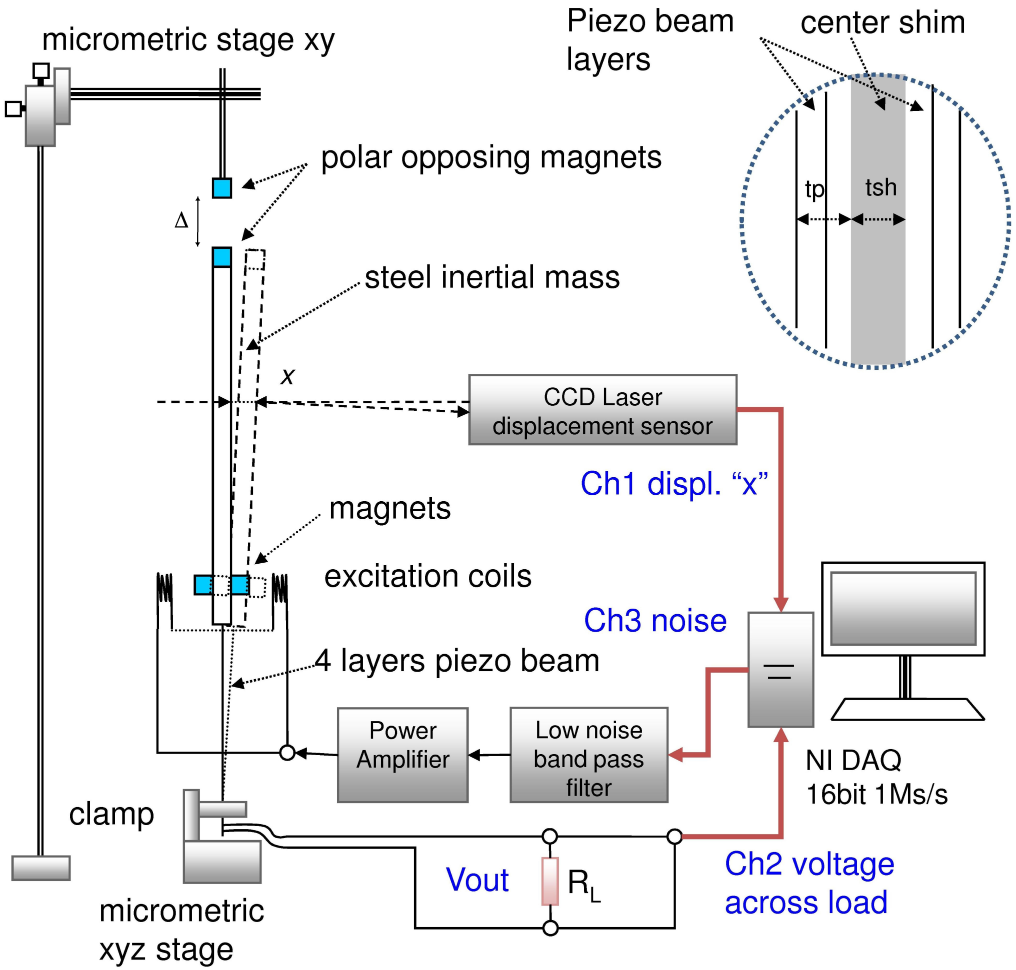




b)



# Noise energy harvesting

NON-Linear mechanical oscillators




$$m\ddot{x} = -\frac{dU(x)}{dx} - \gamma\dot{x} - K_V V + \zeta_z$$

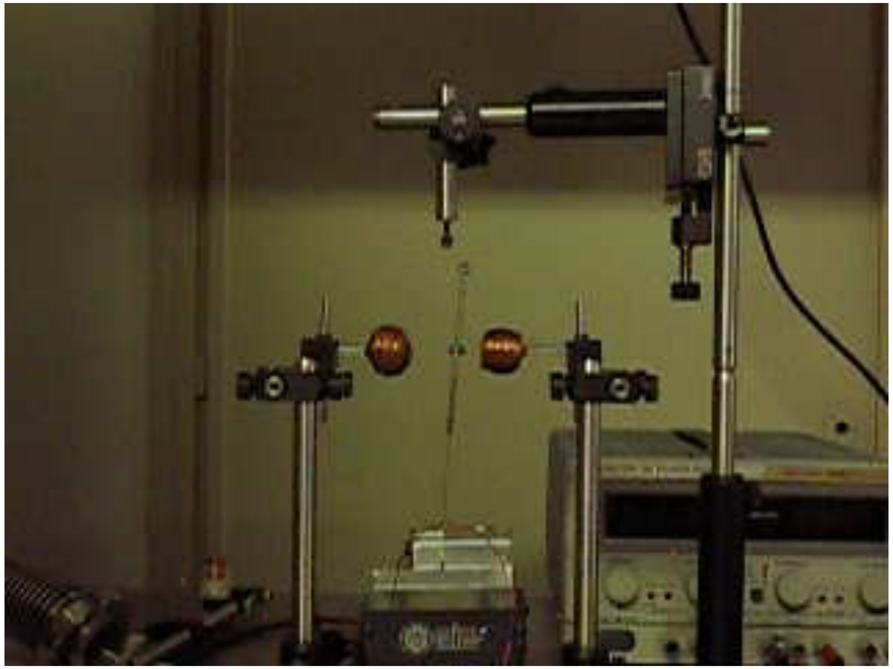
$$\dot{V} = K_c\dot{x} - \frac{1}{\tau_p}V$$

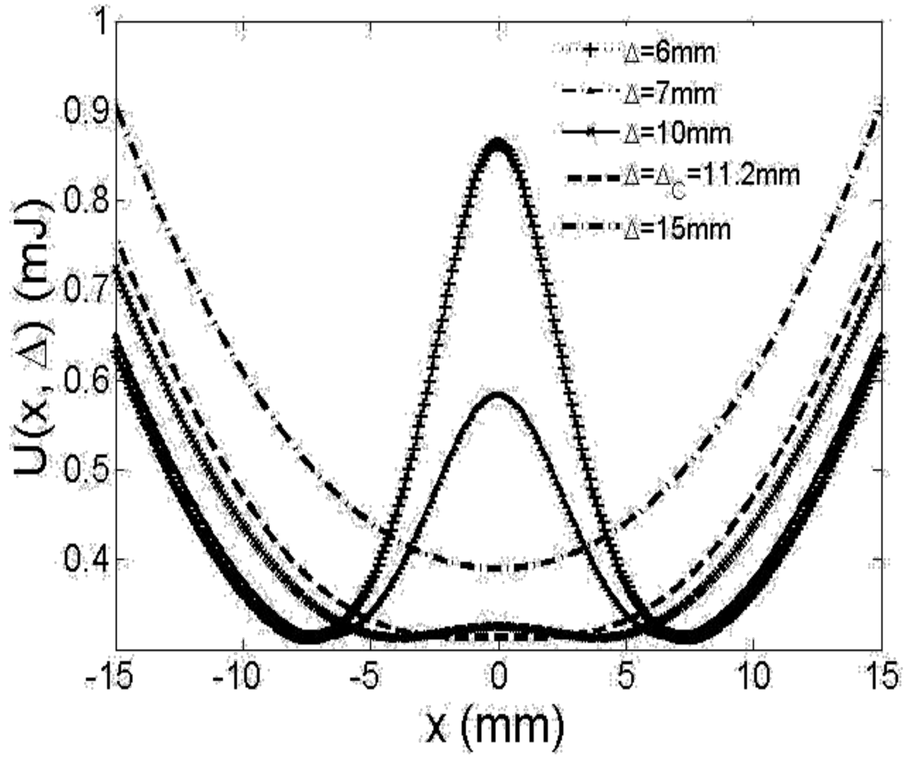
with

$$U(x) = kx^2 + (ax^2 + b\Delta^2)^{-3/2}$$

# Noise energy harvesting

NON-Linear mechanical oscillators



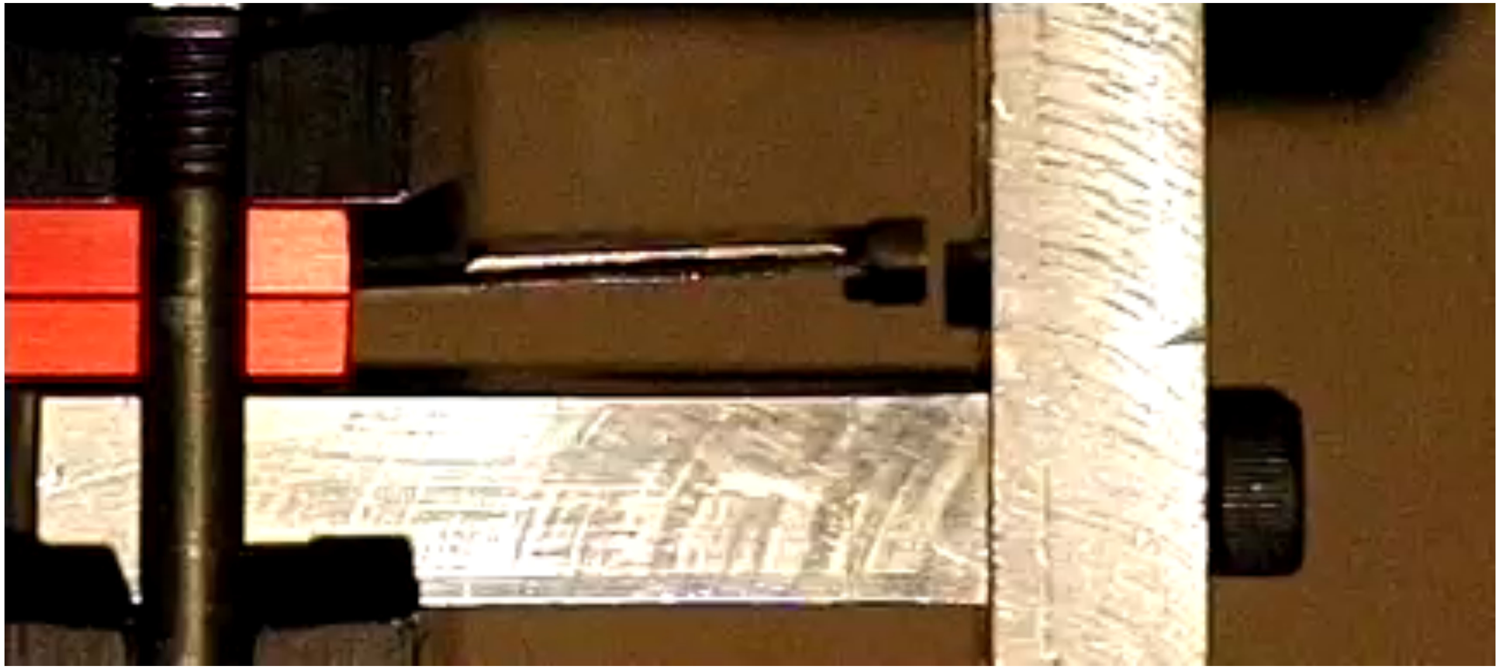



http://www.nipslab.org/node/1676

Nonlinear Energy Harvesting, F. Cottone; H. Vocca; L. Gammaitoni
**Physical Review Letters**, 102, 080601 (2009)

# Noise energy harvesting

NON-Linear mechanical oscillators



Nonlinear Energy Harvesting, F. Cottone; H. Vocca; L. Gammaitoni
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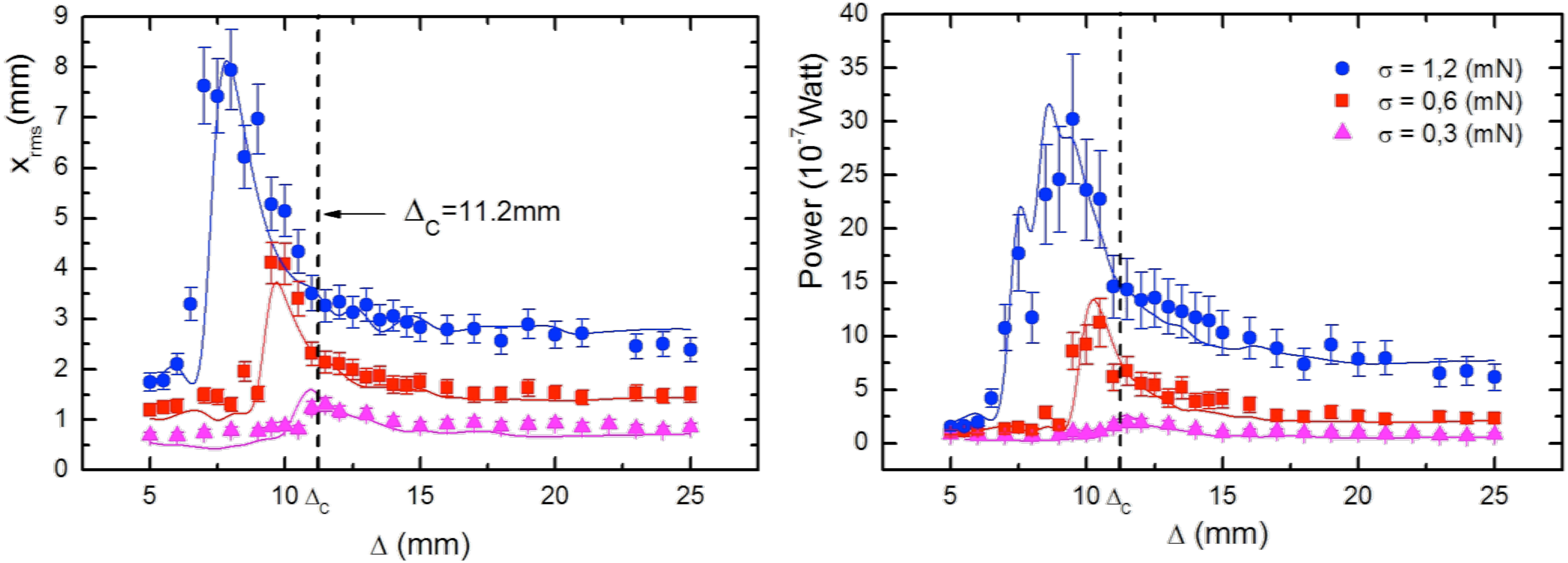
# Noise energy harvesting

NON-Linear mechanical oscillators



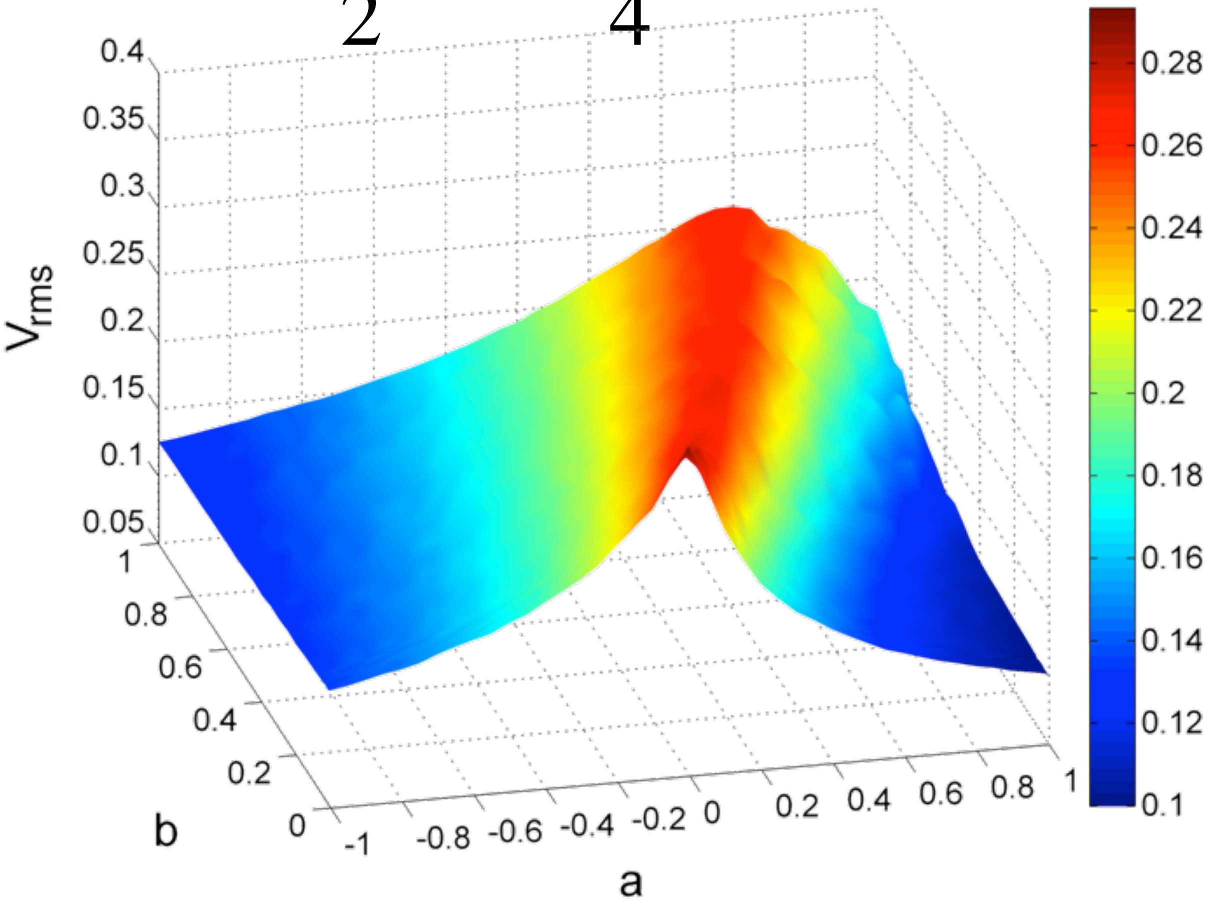

Nonlinear Energy Harvesting, F. Cottone; H. Vocca; L. Gammaitoni , Physical Review Letters, 102, 080601 (2009)

# Noise energy harvesting

Non-linear systems

$$U(x) = -\frac{1}{2}ax^2 + \frac{1}{4}bx^4$$

Duffing potential

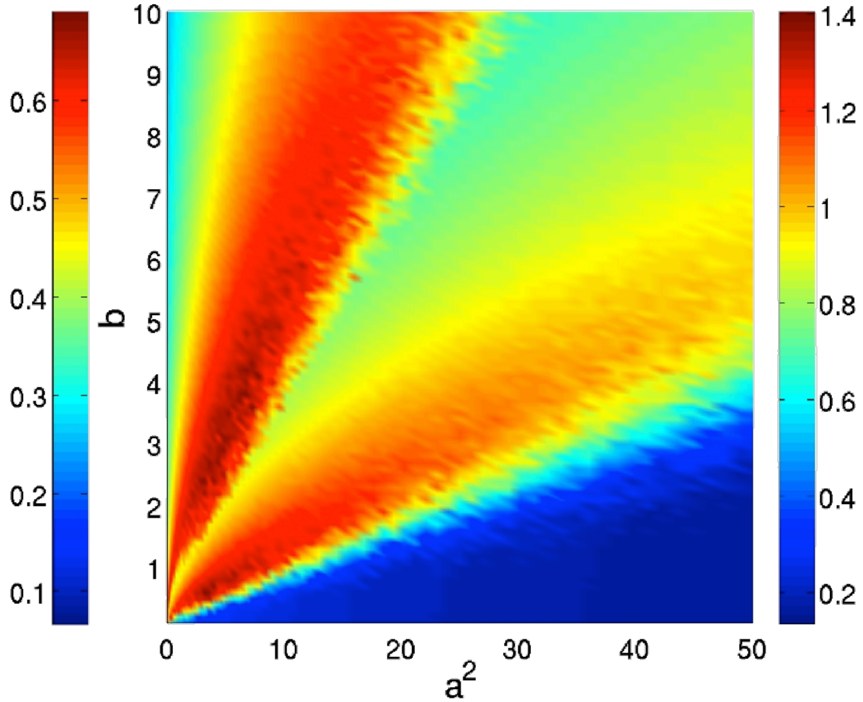



L. Gammaitoni, I. Neri, H. Vocca, Appl. Phys. Lett. 94, 164102 (2009)

# Noise energy harvesting

Non-linear systems

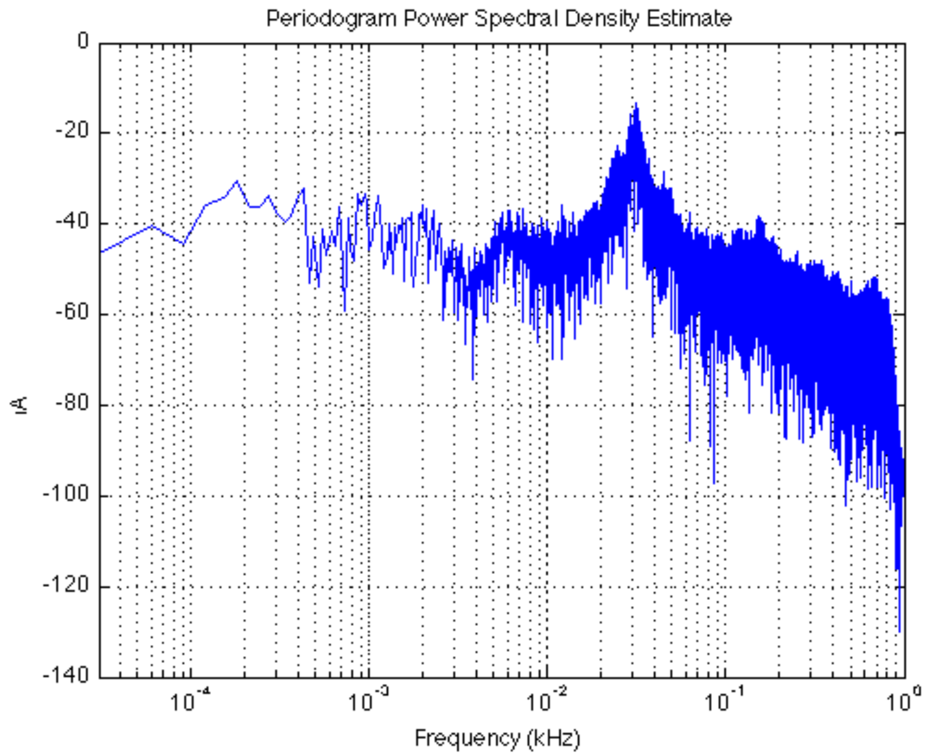
$$U(x) = -\frac{1}{2}ax^2 + \frac{1}{4}bx^4$$ Duffing potential

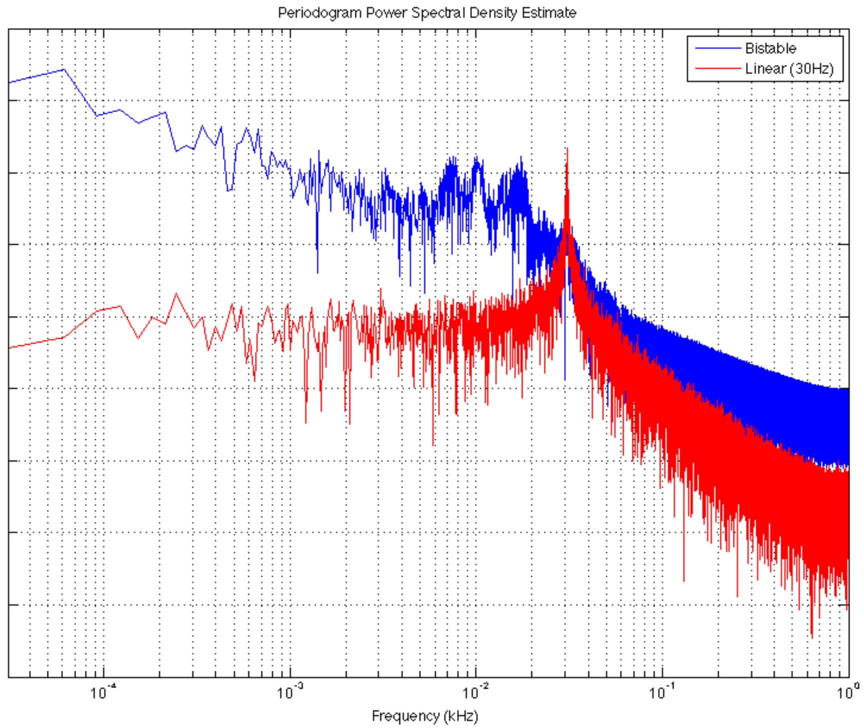


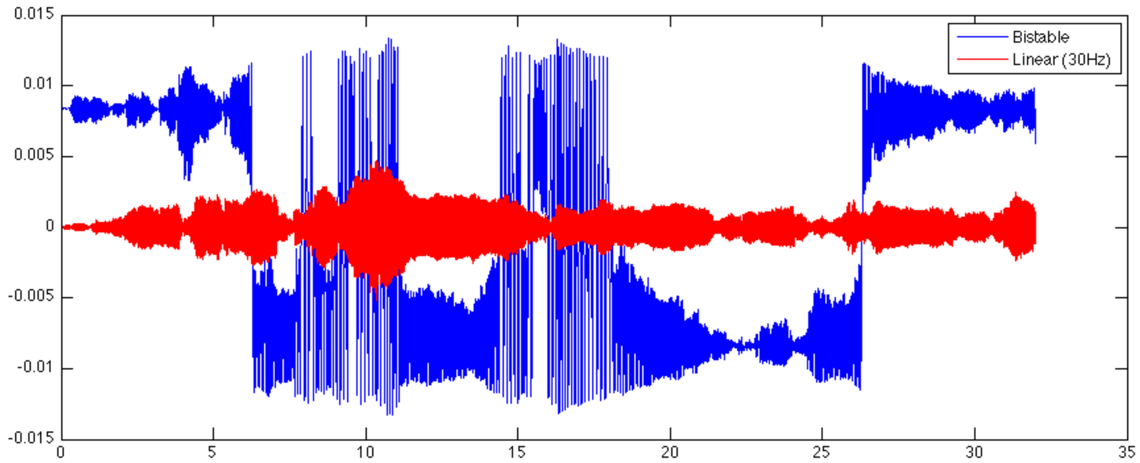

$$b_{MAX} = \frac{a^2}{4D\ \log(\tau_p)}$$

L. Gammaitoni, I. Neri, H. Vocca, Appl. Phys. Lett. 94, 164102 (2009)

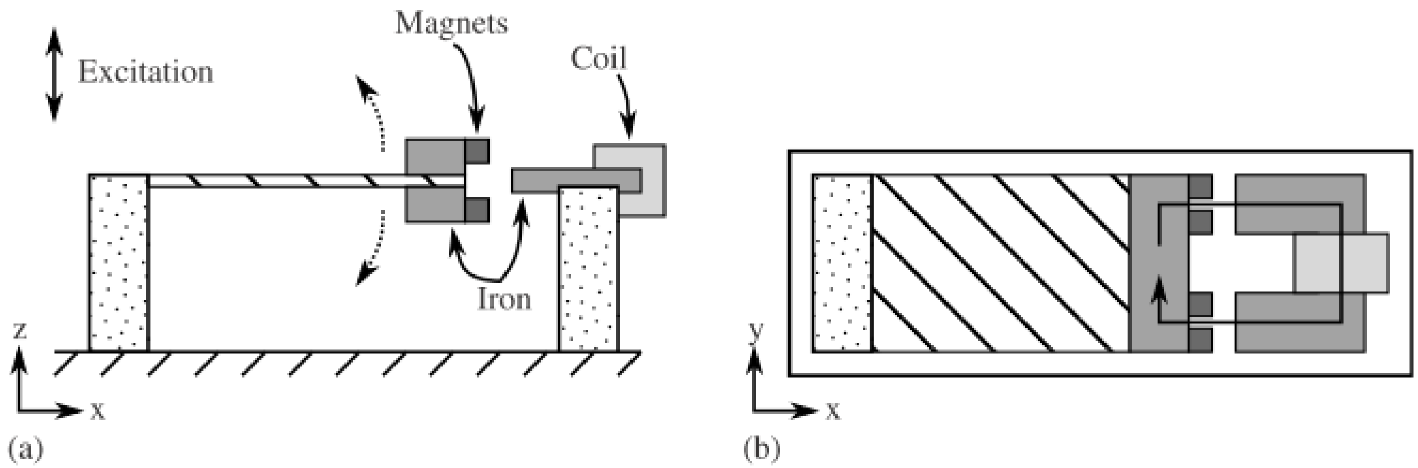
# A compared response



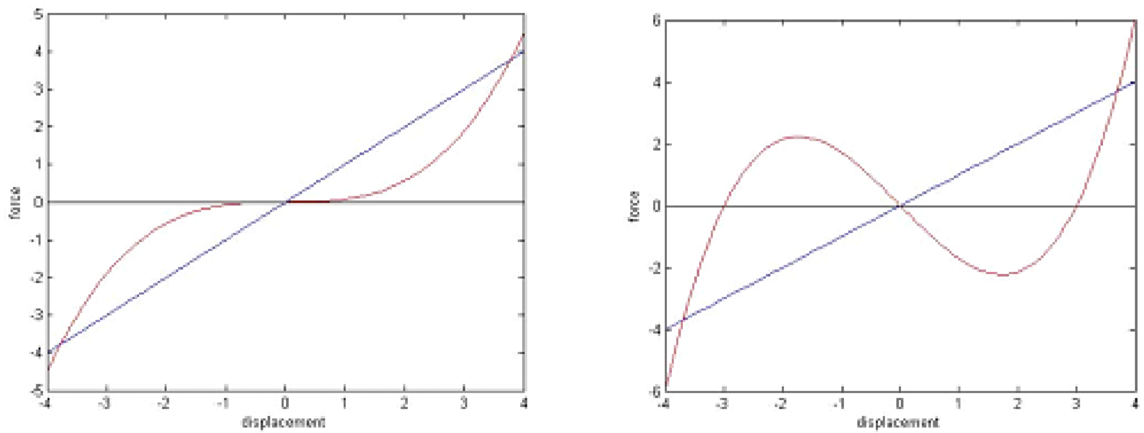
Barton D.A.W., Burrow S.G. and Clare L.R., 2010, "Energy Harvesting from Vibrations with a Nonlinear Oscillator," *Journal of Vibration and Acoustics*, 132, 021009.




$$m\ddot{x} + \left(c + \frac{\theta^2}{R_C + R_L}\right)\dot{x} + kx + \beta x^3 = F\sin(\omega t)$$

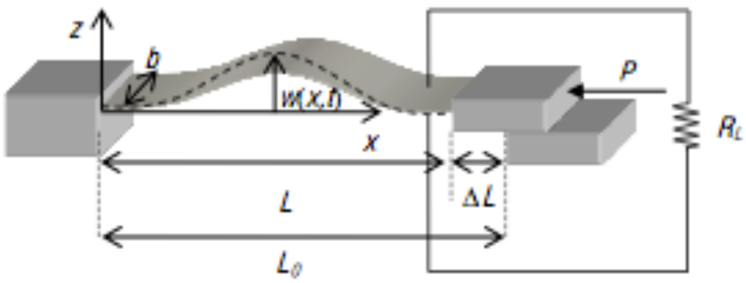
*k* < 0 → hardening

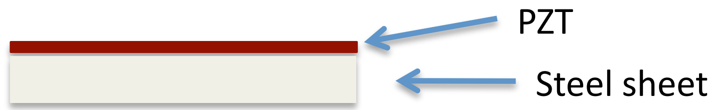
*k* > 0 → bistable

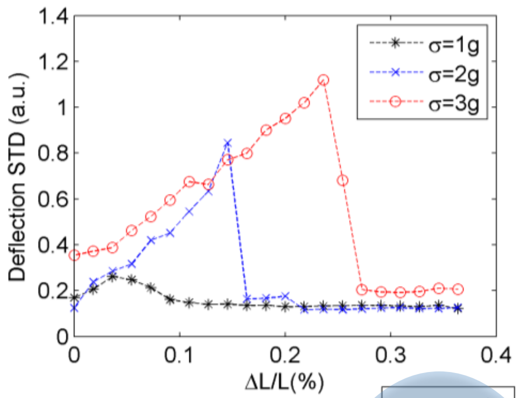


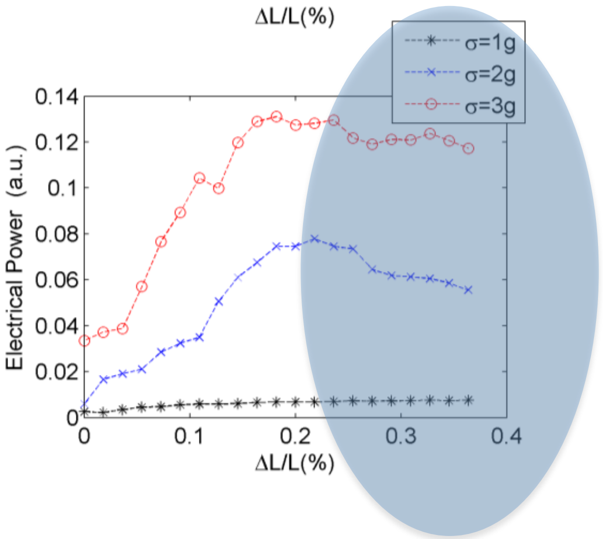

After non-dimensionalization: $\ddot{x} + 2\xi_{eff}\dot{x} + x + \beta x^3 = \Gamma\sin(\omega t)$

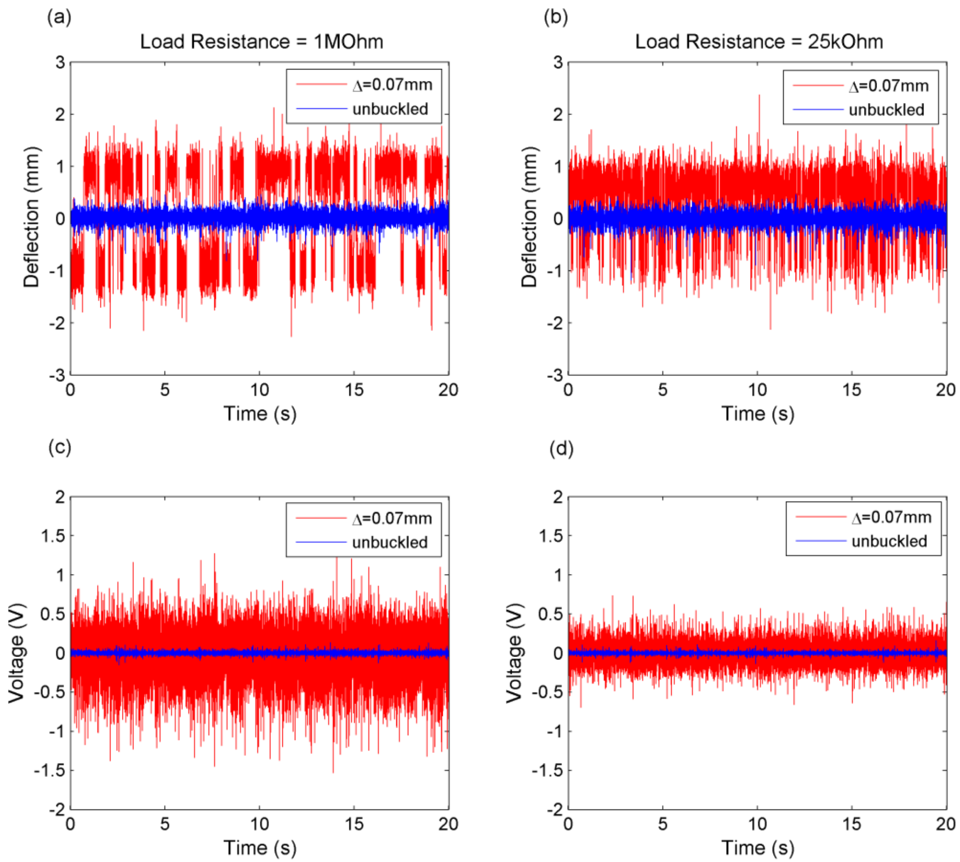
# The buckled beam



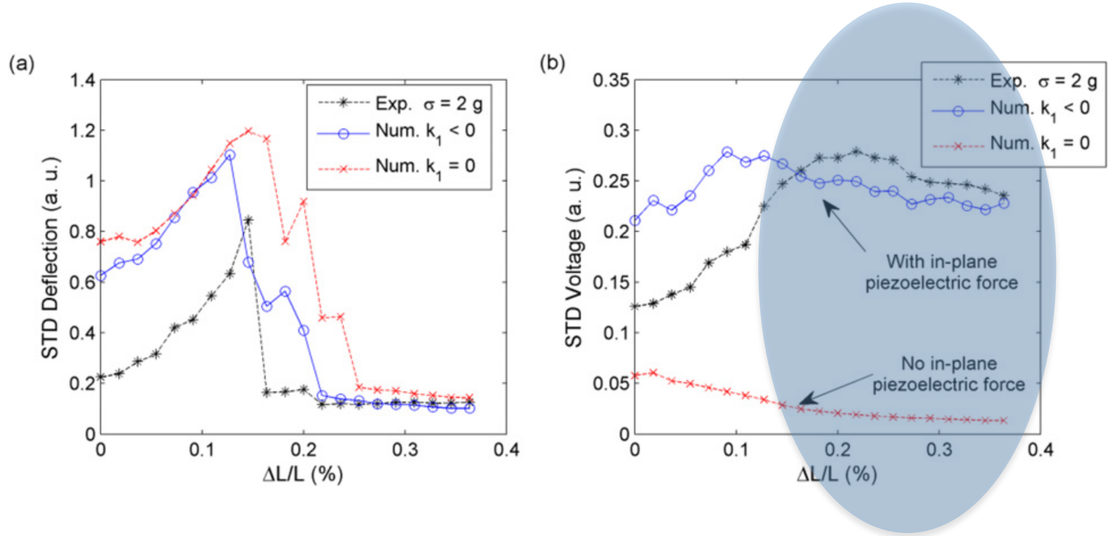




**Piezoelectric buckled beams for random vibration energy harvesting**
*F Cottone, L Gammaitoni, H Vocca, M Ferrari and V Ferrari*
**Smart Materials & Structures 21 (2012)**

# The buckled beam




The governing equations are:

$$m\ddot{x} + \gamma\dot{x} + k_3 x^3 + (k_2 + k_1 V)x - k_0 V = \zeta$$

$$\tfrac{1}{2} C_p \dot{V} + \frac{V}{R_L} = k_1 x\dot{x} - k_0 \dot{x}$$

where $k_2$ and $k_3$ are the linear and non-linear stiffness, $k_0$ is the piezoelectric coupling factor and $k_1$ is the in-plane piezoelectric force factor.

The conservative force is Duffing-like:

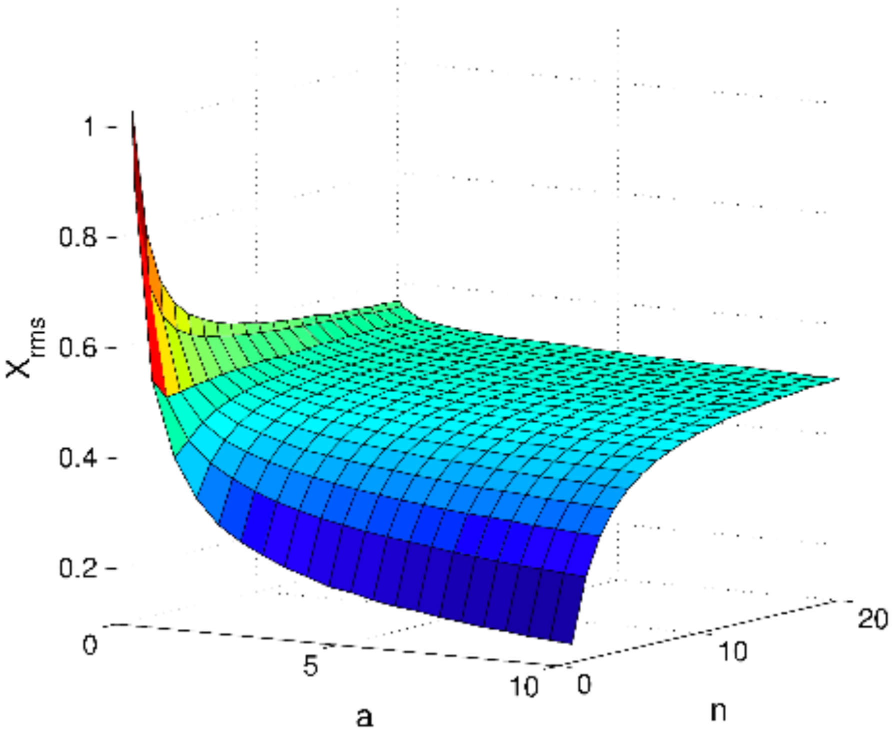
$$U(x) = \tfrac{1}{4} k_3 x^4 + \tfrac{1}{2}(k_2 + k_1 V)x^2$$

where the linear stiffness parameter is a function of the output voltage

# Noise energy harvesting

**Only bistability???**

A more general monostable potential… $U(x) = ax^{2n}$ with $a > 0$, $n = 1,2,\ldots$




In an exponentially correlated noise with correlation time t:

$$\langle \xi(t)\xi(t_1) \rangle = \sigma^2 e^{-\frac{|t-t_1|}{\tau}}$$

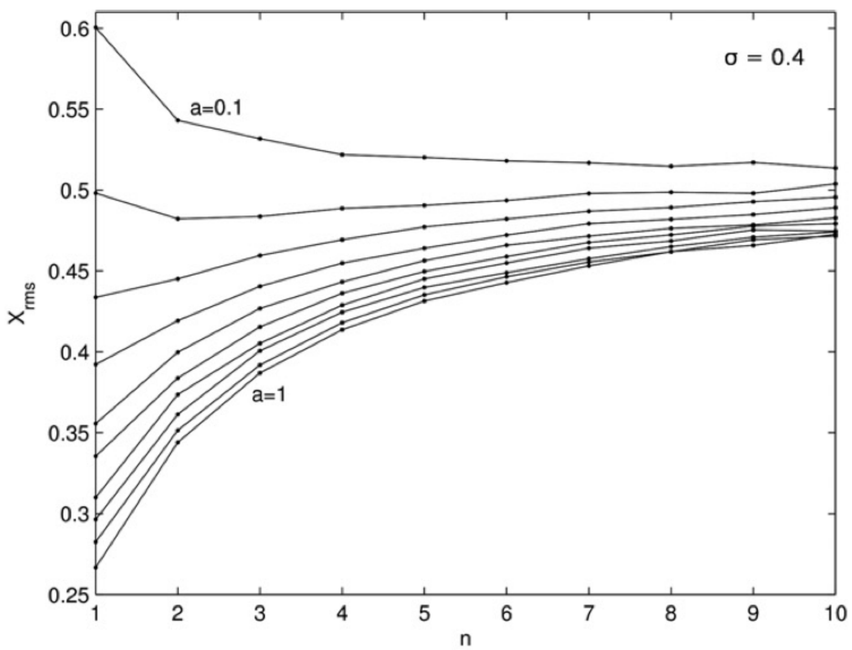
There exists a threshold amplitude $a_{th}$:

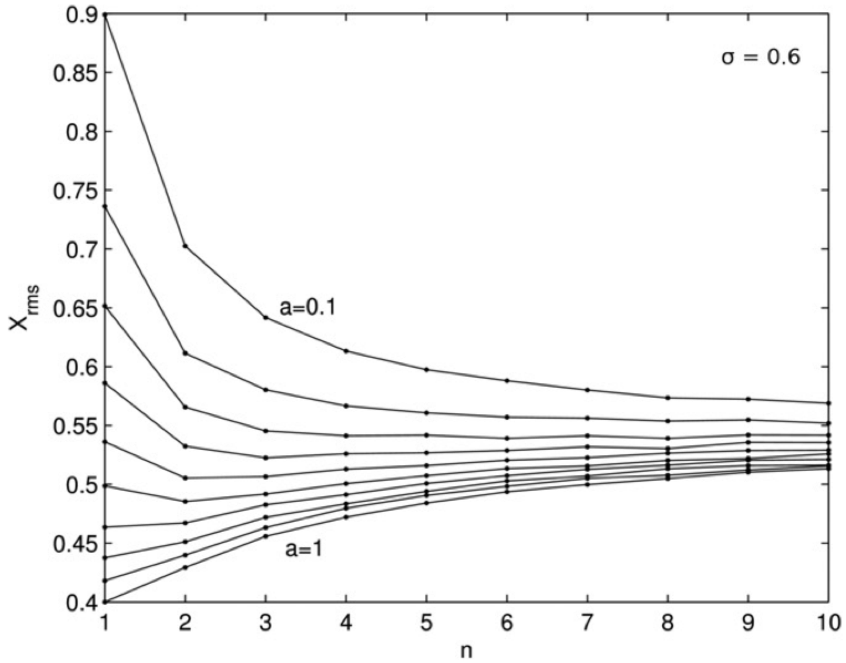
$$a_{th} \approx \frac{D}{4} = \sigma^2 \tau$$

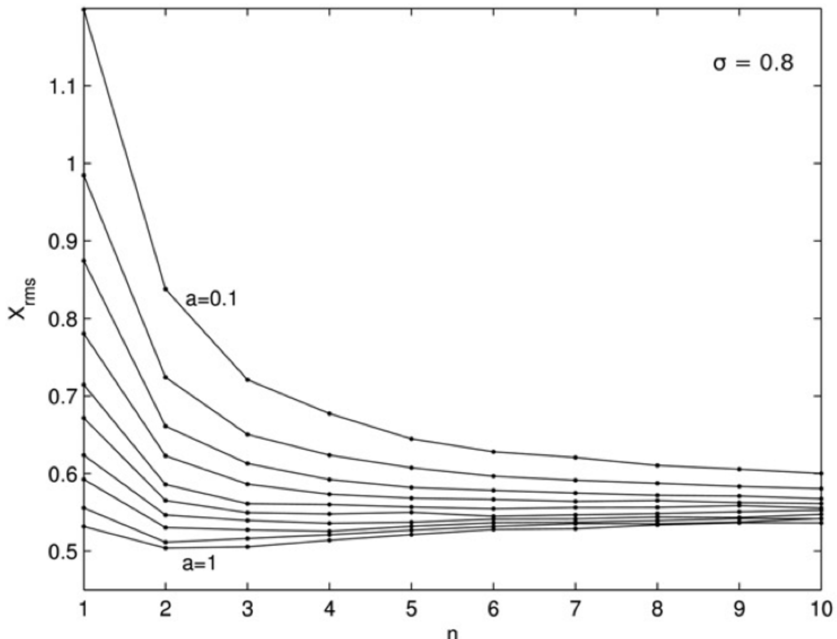
Above which the nonlinear system outperforms the linear one.

L. Gammaitoni, I. Neri, H. Vocca, Appl. Phys. Lett. 94, 164102 (2009)

# Varying the noise amplitude



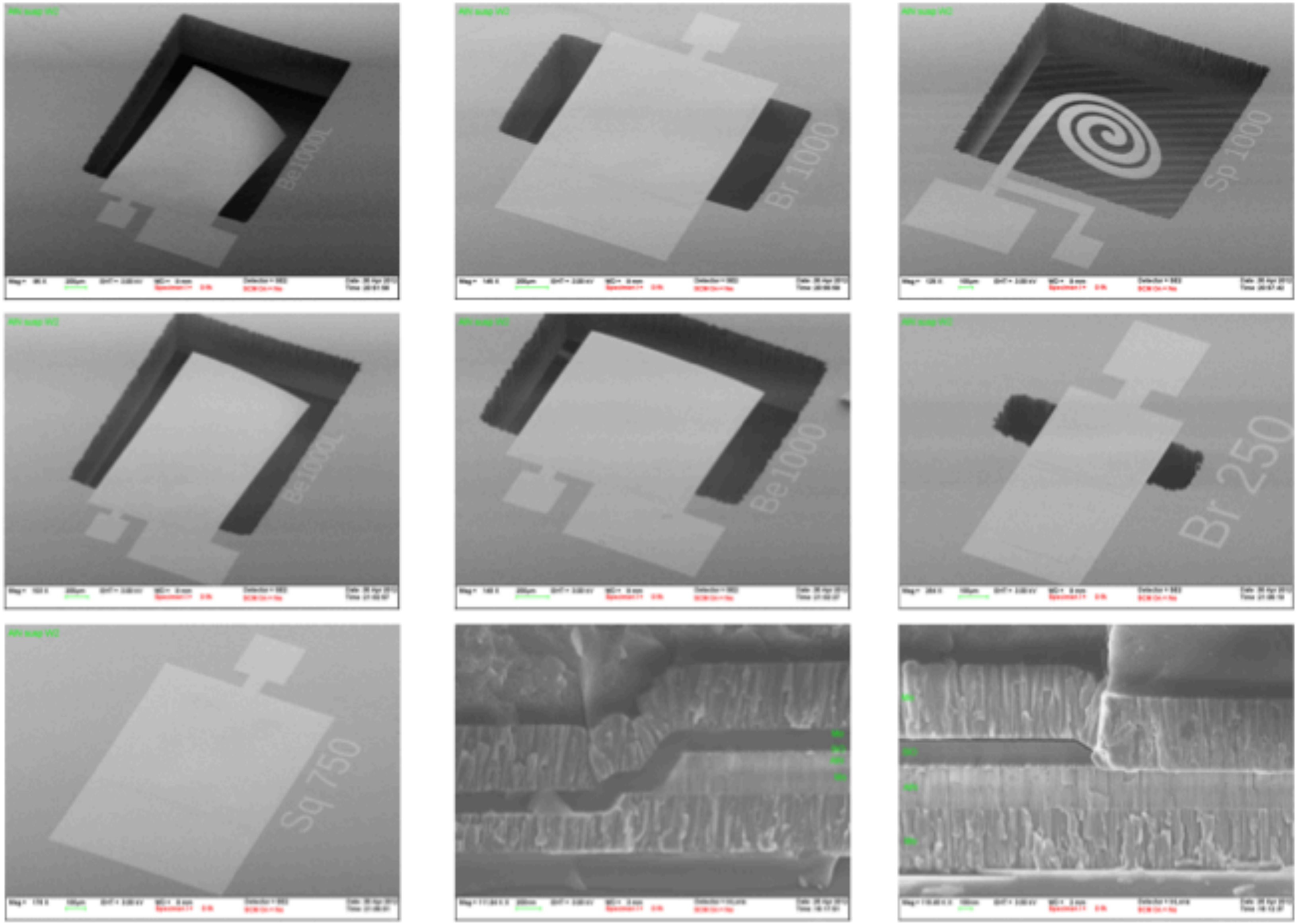



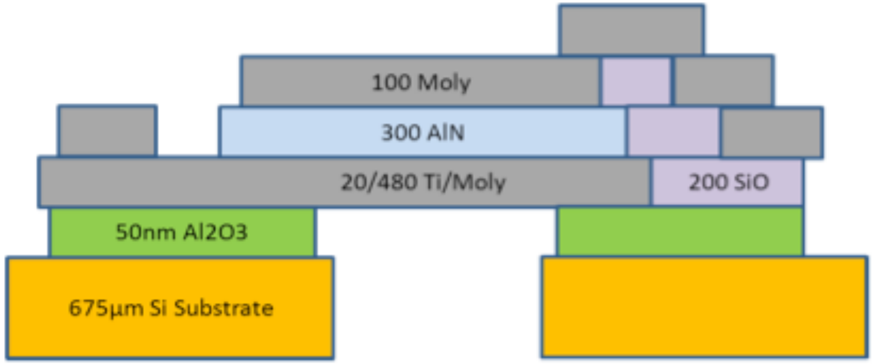

Once **σ** and **a** are fixed the choice of a linear (n = 1) or nonlinear potential (n ≥ 2) can be made in order to maximize $\boldsymbol{x_{rms}}$ and consequently the power obtained at the device output.

L. Gammaitoni I Neri and H Vocca, Chem. Phys. (2010), doi:10.1016/j.chemphys.2010.08.012

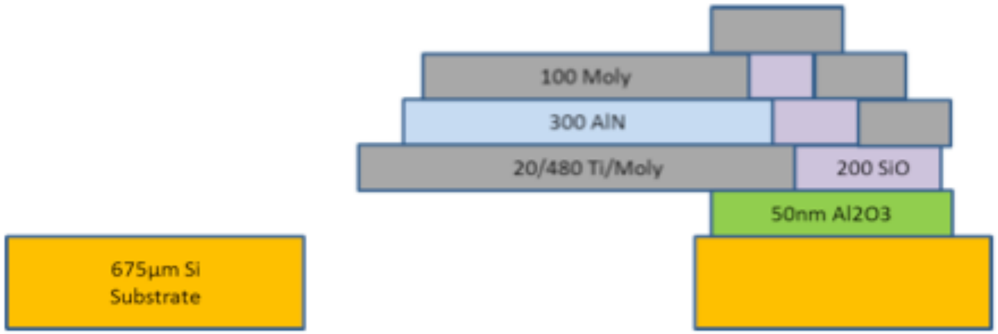
# Nonlinear membranes, beams and … from VTT – Helsinki (Fi)



# Nonlinear membranes and beams from VTT




*Cross section of a membrane harvester shown schematically.*




*Cross section of a beam device supported from only side.*

# Some Videos

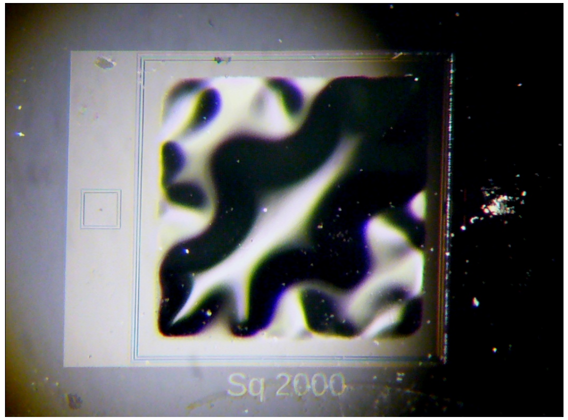
Be 2000- 101 KHz

Br 750- 88 KHz

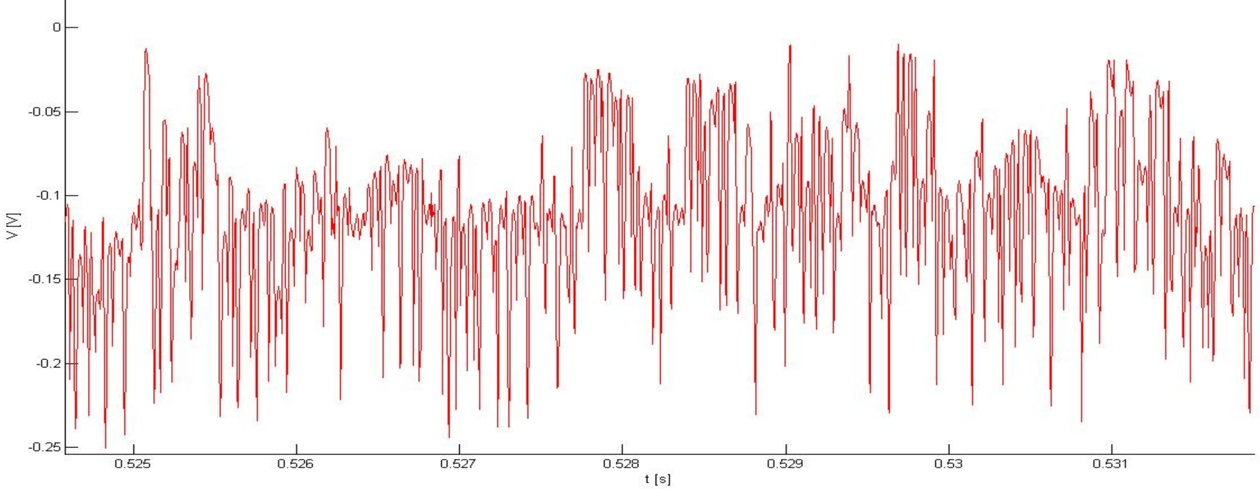
Br 750- 56 KHz

Rn 1000- 59KHz

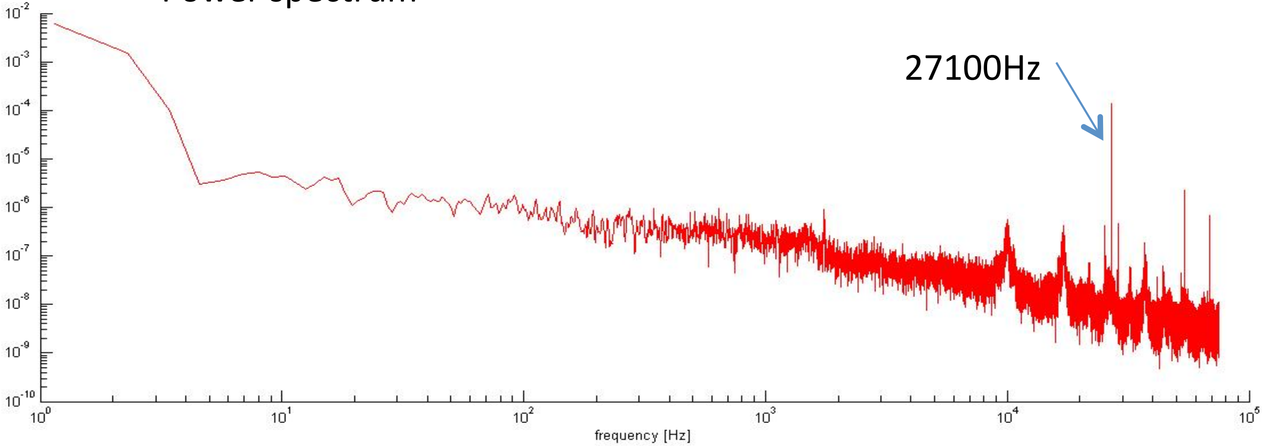
# High frequency measurements




Time serie




Power spectrum

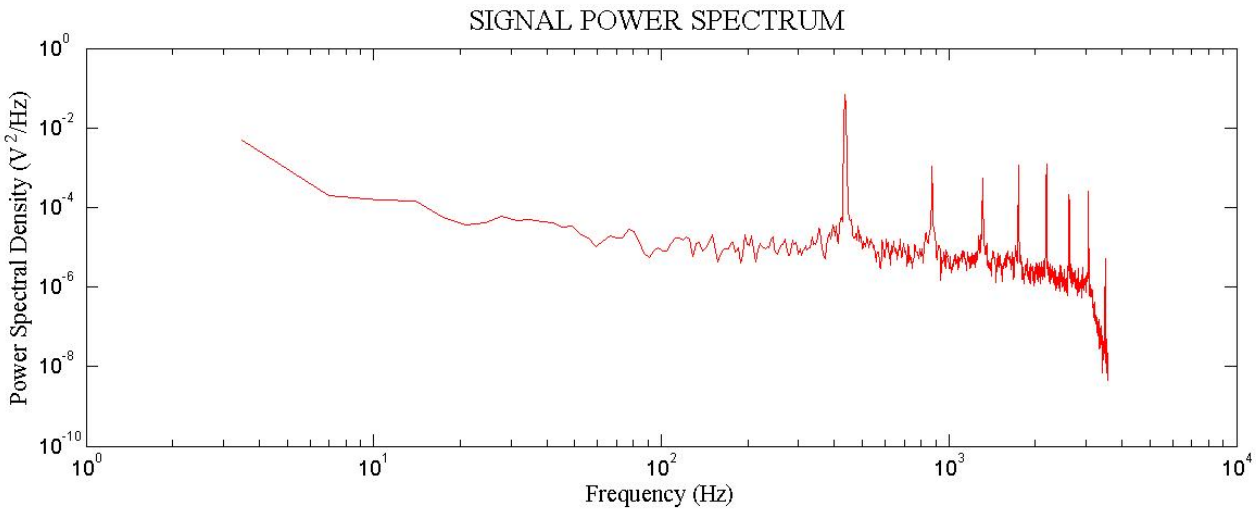



Sine waveform:
10 V-27100 Hz

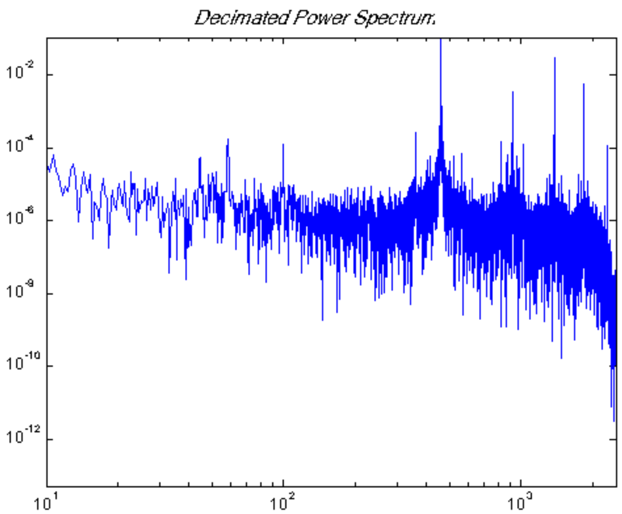
# Statistics for linear systems

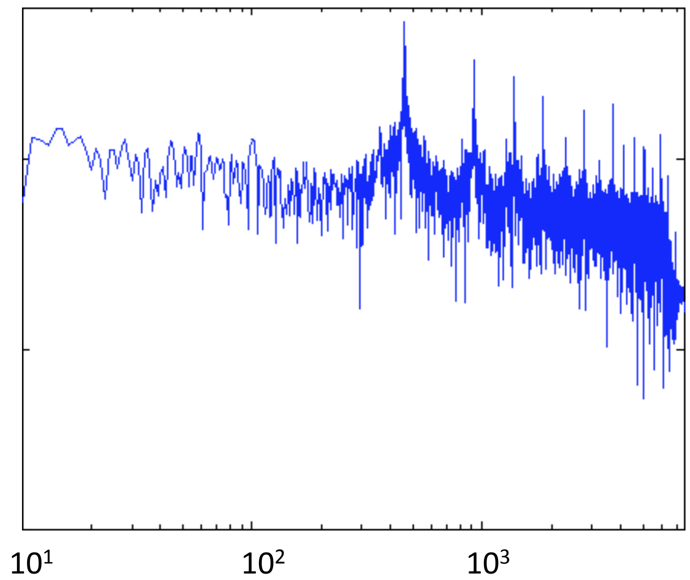
- "1D" Statistics: (2nd Order Cumulants, 1st Order Spectra)
  - Correlation: $C_{xy}(t) = \int_{-\infty}^{\infty} x(\tau)\, y(t+\tau)\, d\tau \quad \Leftrightarrow \quad X(f)\, Y^*(f) = S_{xy}(f)$
  - Power Spectral Density: $C_{2x}(t) \quad \Leftrightarrow \quad X(f)\, X^*(f) = S_{2x}(f)$
  - Coherence: $\mathrm{C}_{xy}(f) = \dfrac{S_{xy}(f)}{\sqrt{S_{2x}(f)\, S_{2y}(f)}}$
    - Tells us power and phase coherence at a given frequency

# DATA ANALYSIS (459 Hz)

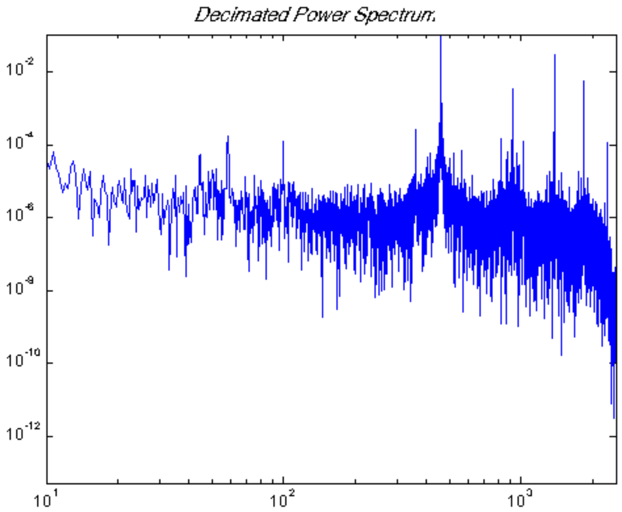



It doesn't tell us much!

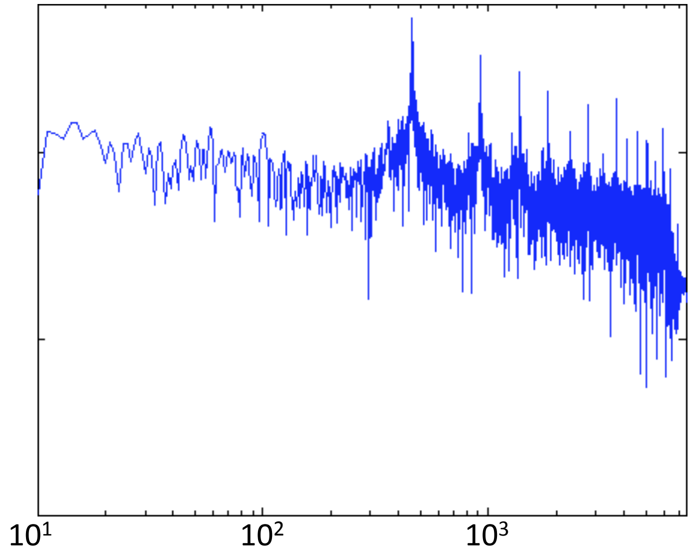


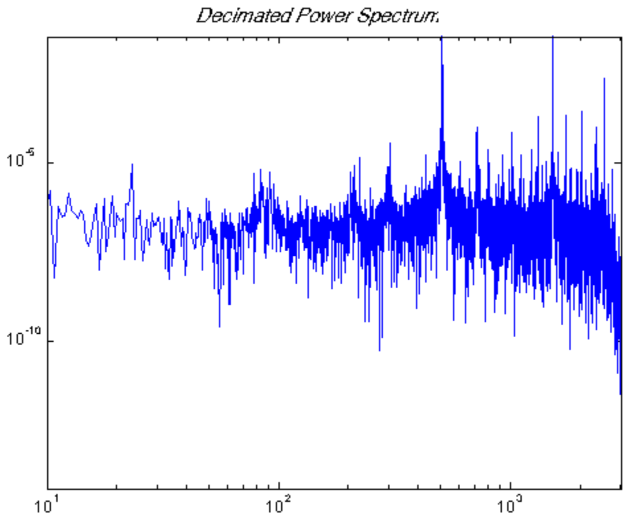



# DATA ANALYSIS

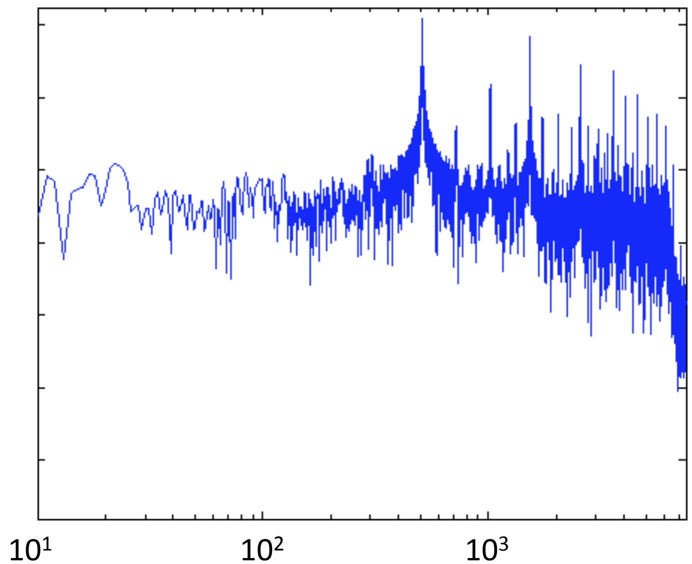



(459 Hz)




(509 Hz)



# Statistics for non-linear systems

- "2D" Statistics: (3rd Order Cumulants, 2nd Order Spectra)

  – Bicumulant:

$$C_{xyz}(t,t') = \int_{-\infty}^{\infty} x(\tau)\, y(t+\tau)\, z(t'+\tau)\, d\tau \quad \Leftrightarrow \quad X(f_1)\, Y(f_2)\, Z^*(f_1+f_2) = S_{xyz}(f_1,f_2)$$

  – Bispectral Density:

$$C_{3x}(t) \quad \Leftrightarrow \quad X(f_1)\, X(f_2)\, X^*(f_1+f_2) = S_{3x}(f_1,f_2)$$

$$S_{3x}(f_1,f_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} C_{3x}(m,n) e^{2\pi i(f_1 m + f_2 n)} dm dn$$

  – Bicoherence:

$$\mathbf{C}_{xyz}(f) = \frac{S_{xyz}(f_1,f_2)}{\sqrt{S_{xx}(f_1)}\sqrt{S_{yy}(f_2)}\sqrt{S_{zz}(f_1+f_2)}}$$

    - Tells us power and phase coherence at a coupled frequency

# Statistics for non-linear systems

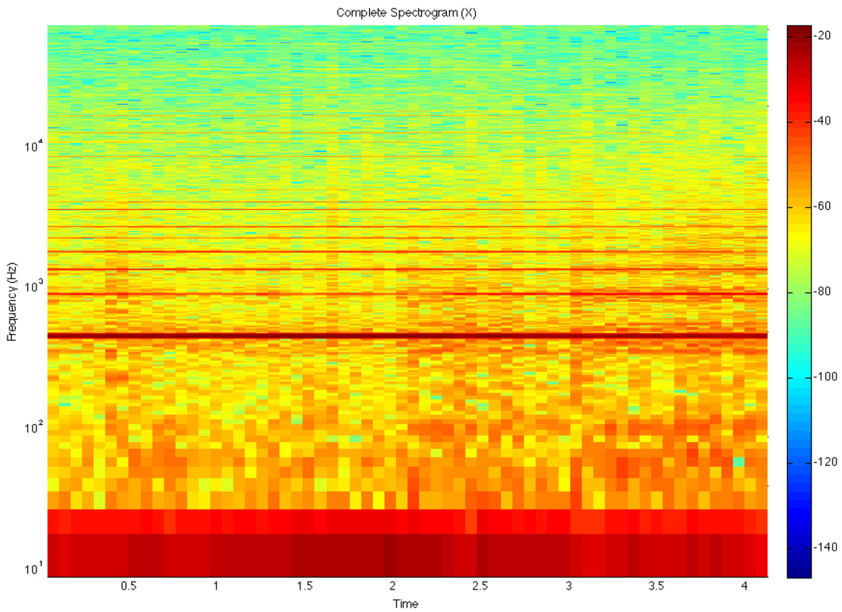
To analize the system linearity bispectrum and bicoherence need to be taken into account:

If $S_{3x} = 0$ the process is Gaussian and linear

If $S_{3x} \neq 0$ the process is not Gaussian and

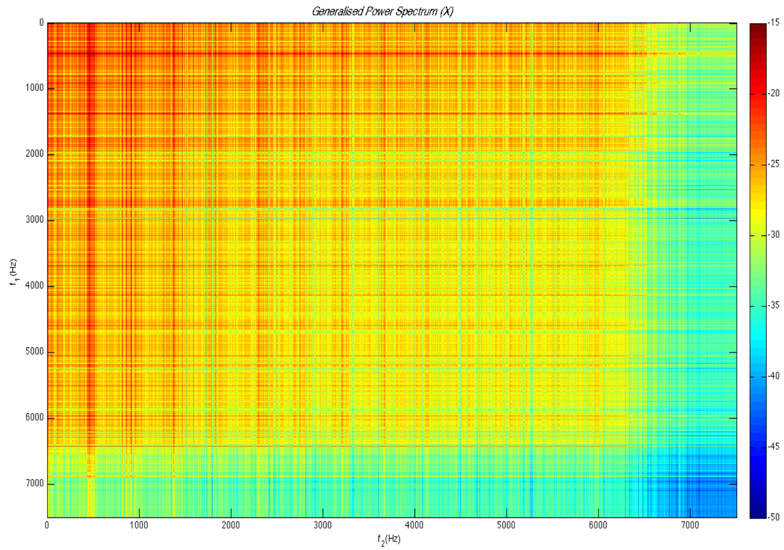
- if $c_{3x}$ is constant - the process is linear
- if $c_{3x}$ is not constant - the process is not linear

# More informations with non-stationary analysis! (459 Hz)

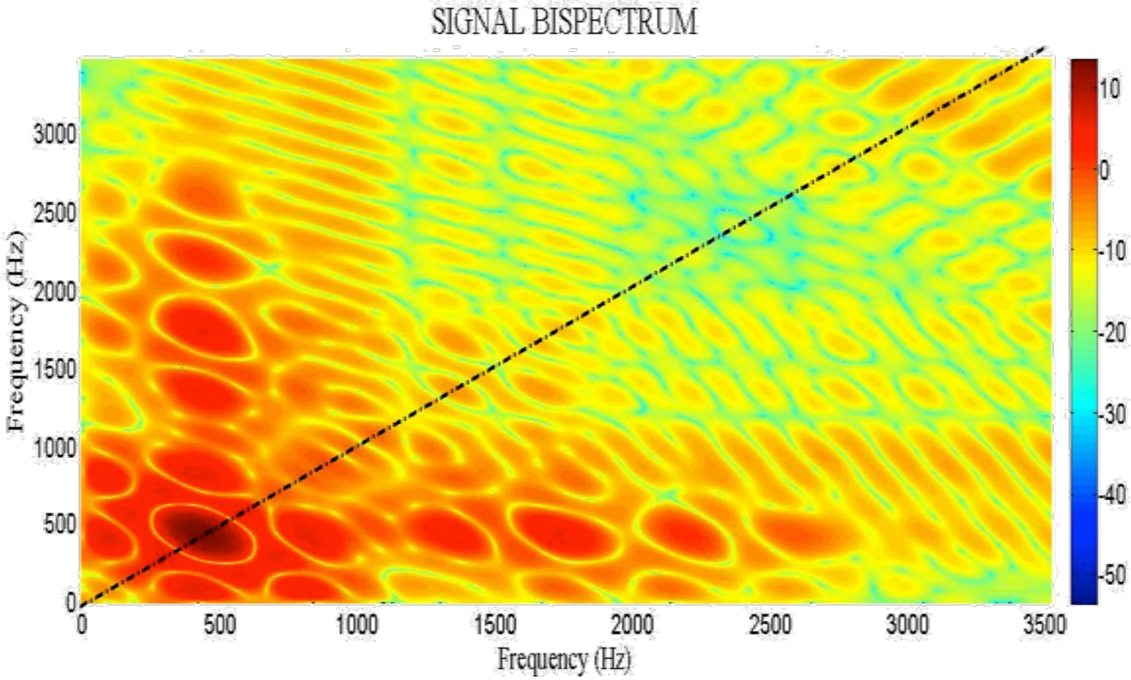



The spectrogram shows nonstationarities and couplings of the forcing signal with lower frequencies.

The Generalised PS confirms this argument.

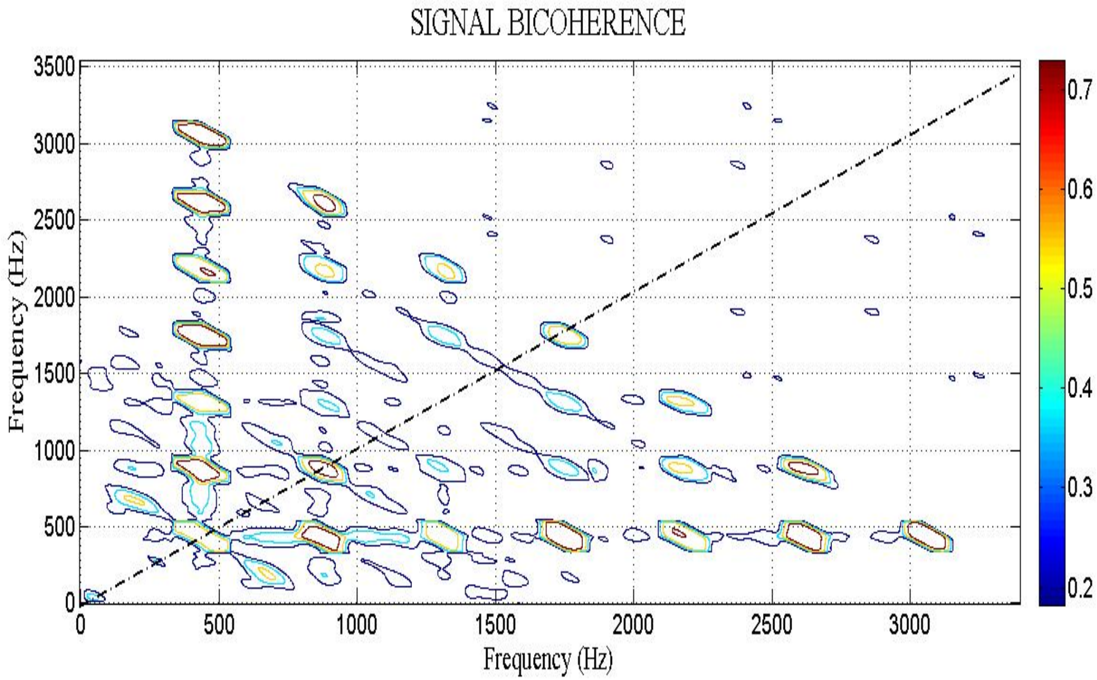


# Bispectrum and Bicoherce for 459 Hz




$$S_{3x}(f_1,f_2)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}C_{3x}(m,n)e^{2\pi i(f_1 m+f_2 n)}\,dm\,dn$$

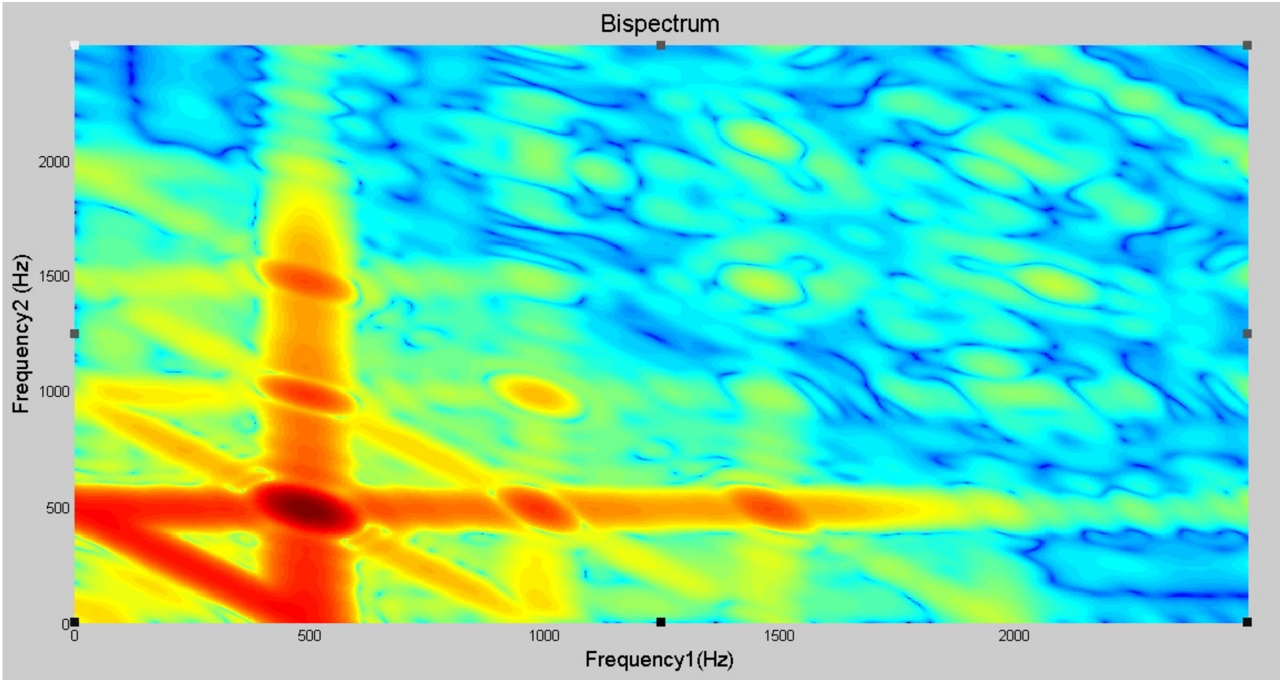
Bispectrum for frequency less then
3.5 KHz.
Information about signal non gaussianity.

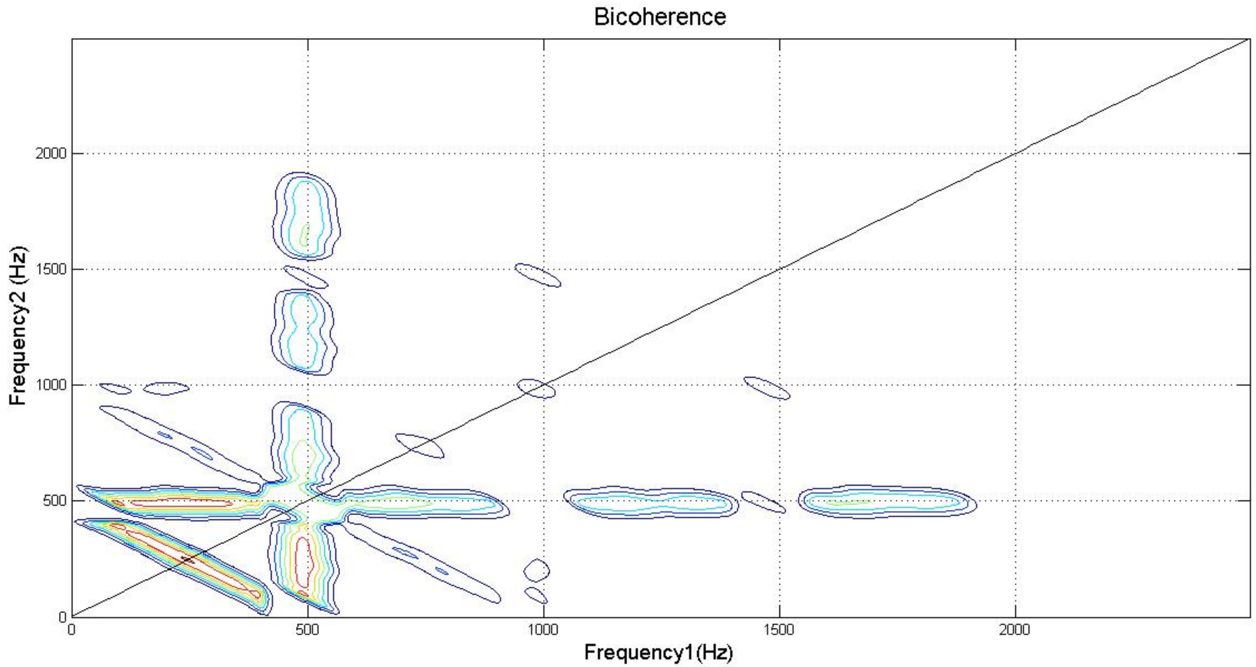



$$\mathbf{C}_{3x}(f)=\frac{S_{3x}(f_1,f_2)}{\sqrt{S_{xx}(f_1)}\sqrt{S_{xx}(f_2)}\sqrt{S_{xx}(f_1+f_2)}}$$

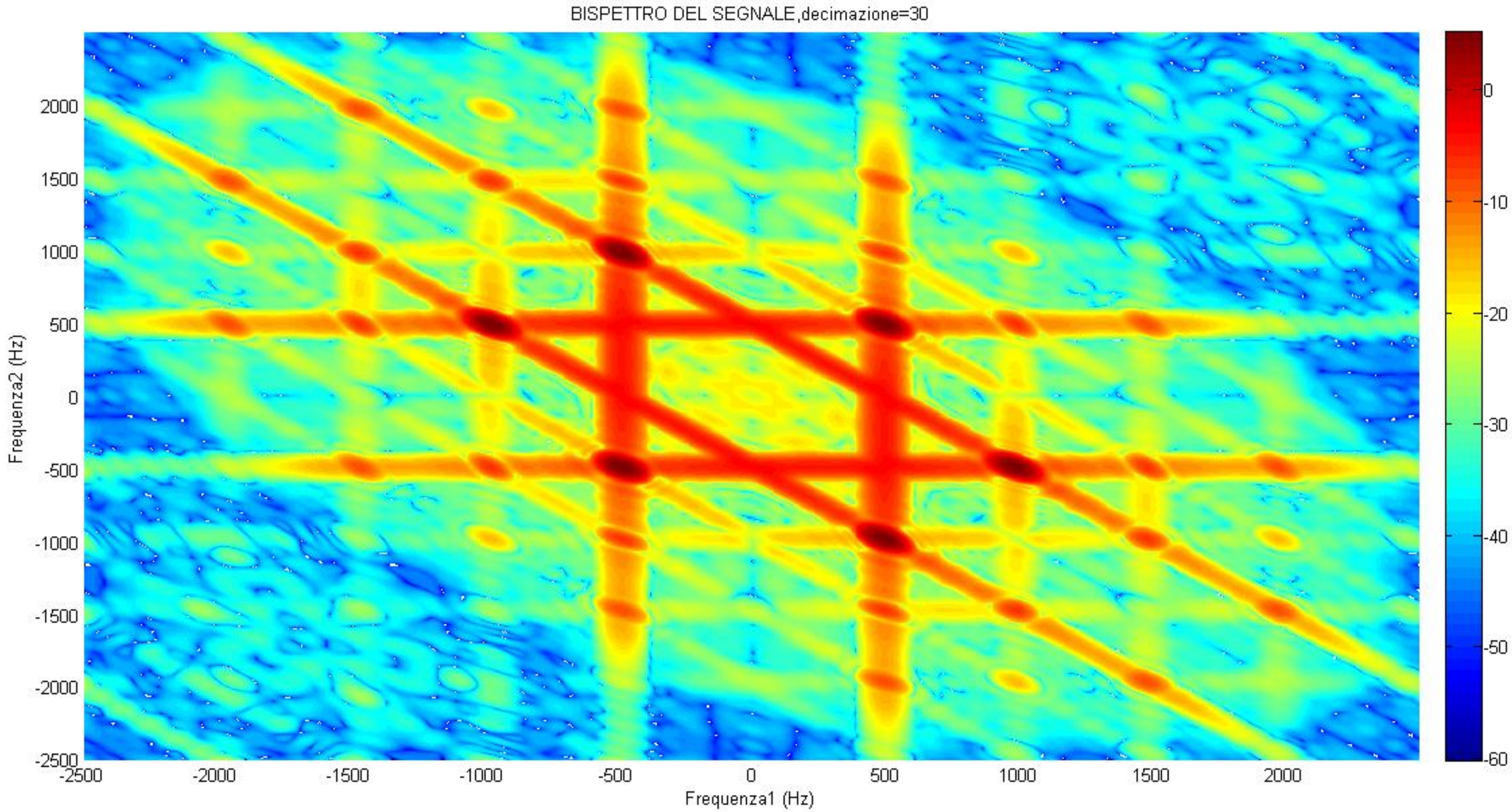
Bicoherence for frequency less then
3.5 KHz. Information about non linearity.
Couplings that cannot be attribuited to the
forcing signal.

# Bispectrum and Bicoherce for 491 Hz



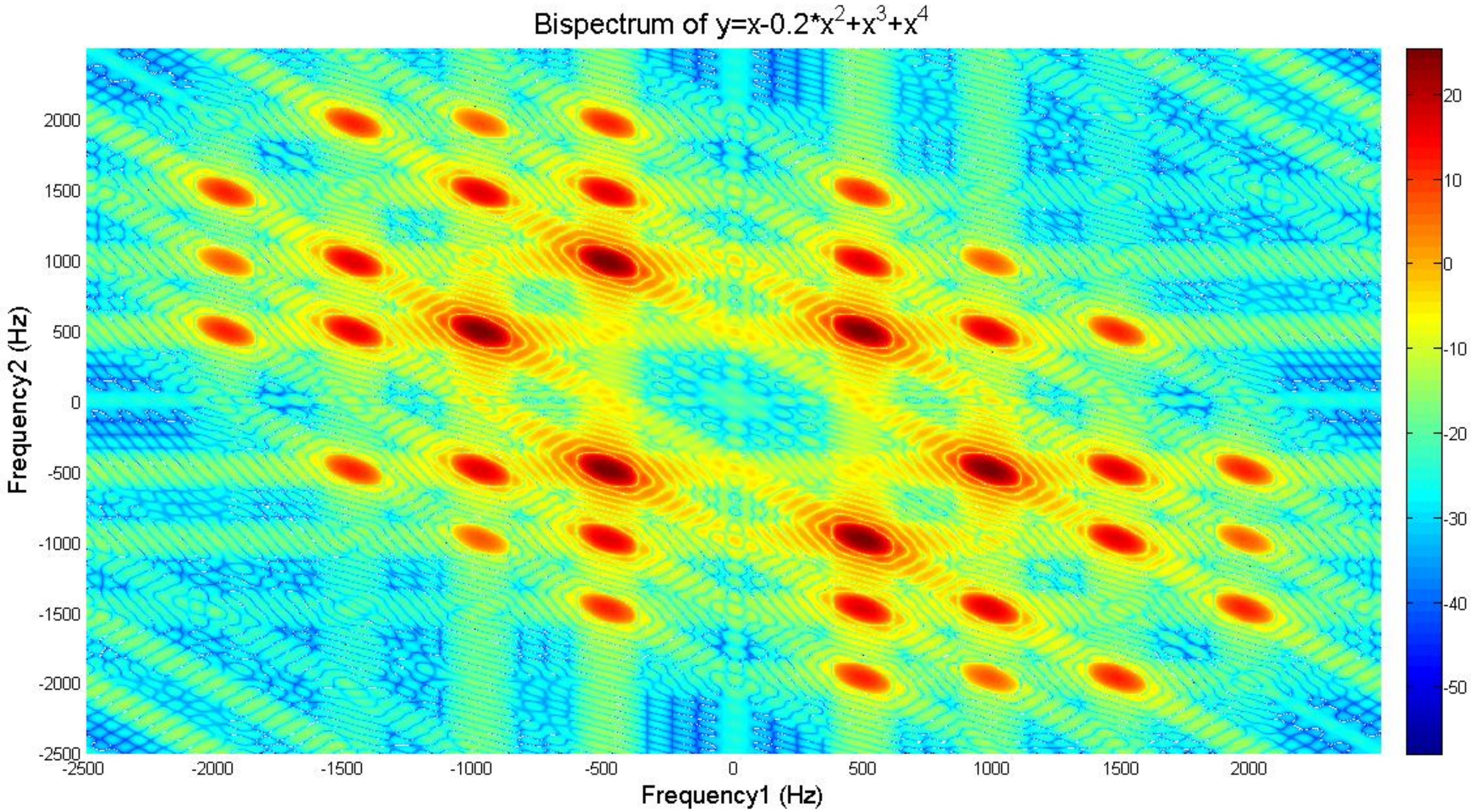



**With nonlinear analysis we can design beautiful tiles…**




Where do these lines come from?

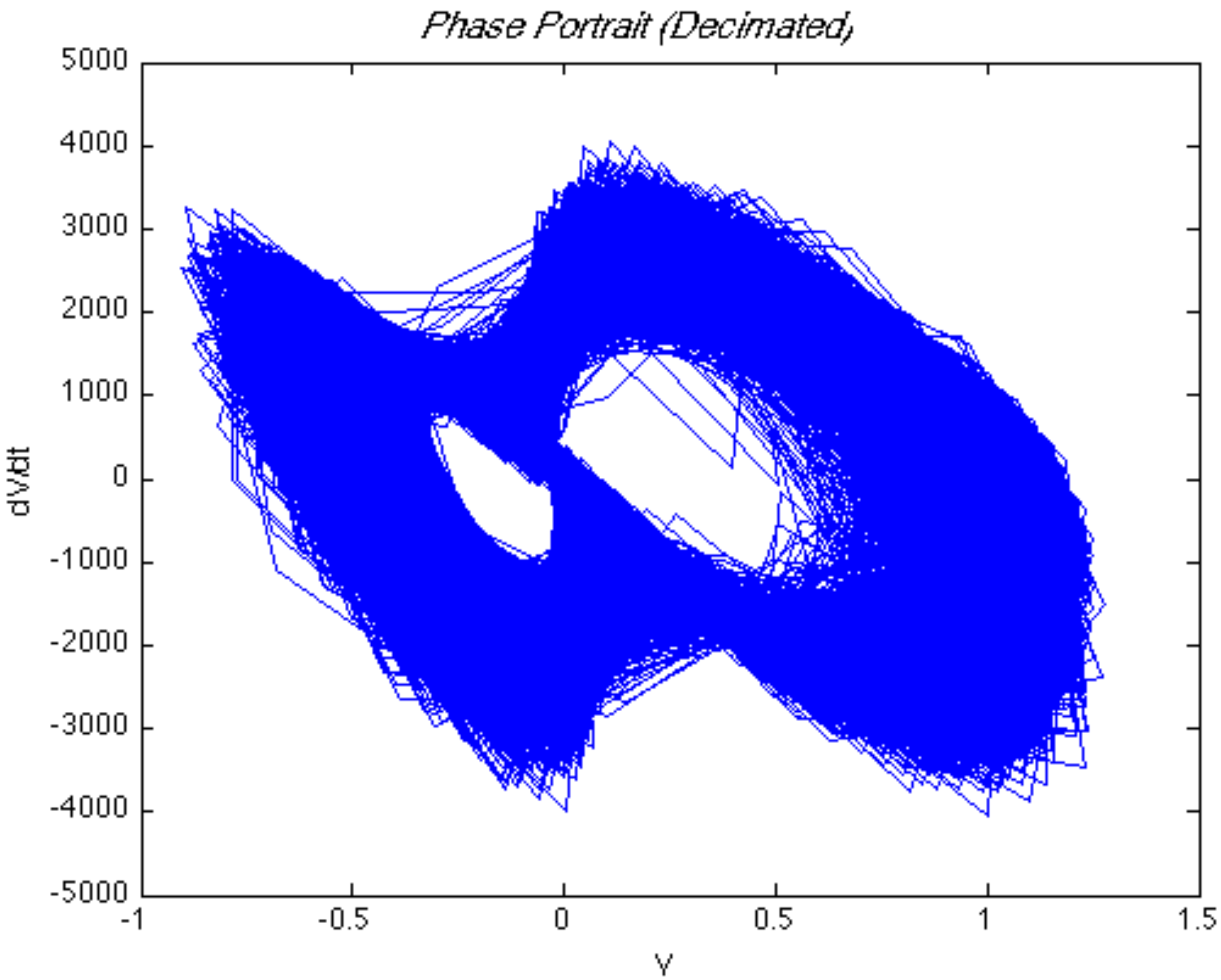
**With nonlinear analysis we can design beautiful tiles…**




The lines are a consequence of the nonlinear coupling of the first three orders!

## *Where the couplings come from?*

There's a clear sign of bistability.




The membrane dynamics show two separted attractors and (probably) a chaothic beaviour.

# *Conclusions*

*Let's try to enjoy the Suprematism!!!*

*It's not easy, but can give some good results…*

*"… nothing is real except feeling... "*
("Suprematism", Part II of *The Non-Objective World*)

# *Acknowledgments*

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*Office of Naval Research (ONR)*